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Original Paper

Excel Files for Newton's Proposition V

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Abstract

Newton in Principia gives us a mathematical method of finding the center of force for a body moving on an ellipse in Proposition V, Problem I. The same thinking can be applied also to the case of a hyperbola and also a parabola, only that in the last case the center of force is at infinite distance. For the first two cases there are 3 cases of possible forces: a) An force proportional to the distance from the center. For ellipse an attractive force for hyperbola a repulsive, b) a force proportional to the inverse of the square of the distance from the left focus, for ellipse an attractive and for hyperbola a repulsive force, c) an attractive force inversely proportional to the square of the distance, inversely proportional to the square of the distance for both the ellipse and the hyperbola. This method when applied to the case of circular orbits for which we can find the center of force with the same method: Newton studied a semicircular orbit with center of force at infinite distance, and the case of a central force whose center is located on the circular orbit or inside the circle studied the case of a spiral orbit. In each the law of the force was derived by using the law of areas.

Keywords

conservation of angular momentum, law of areas, central forces, law of force

1. Introduction

In Newton's words: *There being given, in any places, the velocity with which a body describes a given figure, by means of forces directed to some common center: to find that center* (Newton, n.d.). His proof is simple but also needs some explanation: Newton uses the constancy of angular momentum which he explained in by PROPOSITION I, Theorem. *The areas, which bodies made to move describe by radii drawn to an immovable center of force lie unmoving planes, and are proportional to the times in which they are described* (Newton, 1999).

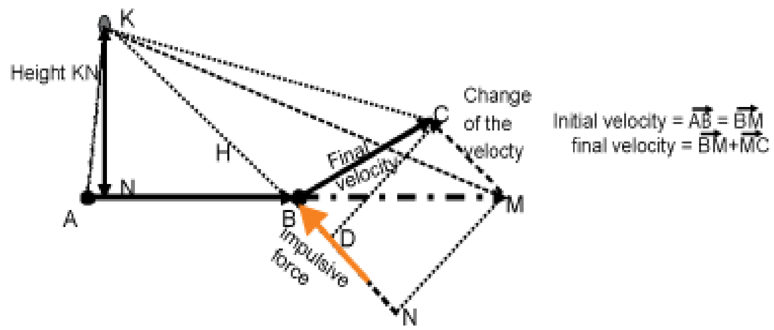

Figure 1. Law of Areas

Actually this proposition is very useful for teaching purposes. The length of the movement in the unit of time is equal to the velocity. If from the center K we draw the radii KA, KB, KC, then the areas KAB, KBC, are equal, since the triangles KAB, KMN have equal bases and equal heights, but also the triangles KBC and KBM have the same base (KB) and heights DC=NM As we can see in Figure 1, the constancy of the areas is valid for central force which for Newton's model is given by impulses along the lines which join the end of the velocity vector to the center of force. As is seen in the Figure 1 the law is valid in the case of a central force.

2. Proposition V, Problem I for Ellipse

In the case of the ellipse we consider as Newton (1981) points *P, Q, R* where we draw the velocity vectors *VP, VQ, VR*. One question that can be asked is what kind of force caused this movement (Figure 2)? The students can check the figure and guess the force. They can think from the size of the 3 velocities. The teacher can move a little bit the points and the students can try to think of where is the center of force. This problem was used in attempts to teach Principia (Alfred, 1964).

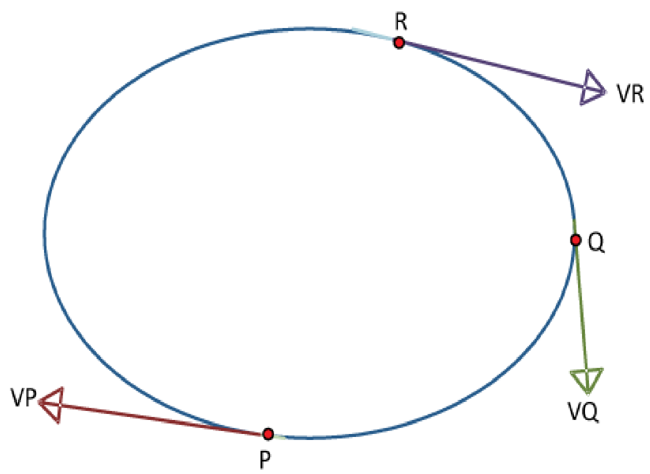

Figure 2. Where is the Center of Force that Produces the Velocities?

It is known from history that Newton was asked by Halley, what he thought the Curve would be that would be described by the Planets supposing the force of attraction towards the Sun to be reciprocal to the square of their distance from it (Newton and Kepler Ex Libris, n.d.). Newton answered that this curve is an ellipse. In the present problem we have an ellipse but the answer is not always a force reciprocal to the square. Newton gave in Principia the proposition which can be answered by either gravitational forces directed towards either one of the foci or by a force proportional to the distance from the center and directed to it (which was proposed by Hooke (Michael, 1994)). The motion around the center will have a constant angular momentum, or the areas described by the radius drawn from the center of the force to the particle will be equal in equal times. To find the center we precede either by calculating with the use of analytic geometry or with Newton's method (the equations from which we derive the center of force, are presented in the end of the paper). We draw the three lines *PT, TQZ, ZR* which touch the curve at the points, P, Q, R, and meet in T and Z (see Figure 3) On the tangents erect the perpendiculars PA, QB, RC, reciprocally proportional to the velocities of the body in the points P, Q, R, from which the perpendiculars were raised. So $A = \frac{k}{VP}, QB = \frac{k}{VQ}, RC = \frac{k}{VR}$. From *A*, *B*, we draw perpendiculars to PA and QB correspondingly. These perpendiculars meet at D. We draw from C a perpendicular to RC which meets BD at E. We draw from D the perpendicular DP" to VP, from D the perpendicular DT" to VQ, and form E the perpendicular ER" to VR.

So we have $VP \cdot PA = VP \cdot DP'' = VQ \cdot BQ = VQ \cdot DT''$ So for point *D* we have $DP'' \cdot VP = DT'' \cdot VQ$. Also for point E we have $VQ \cdot EZ'' = VR \cdot ER''$. These relations resemble the law of areas but actually they are valid only on pairs of velocities. We can repeat the procedure for different values of k. For $k \to 0$ the limiting lines are the tangents at P, Q, R which meet correspondingly at T and Z. If we increase the value of k we get a smaller distance of DE and we can reach the point where the two lines ZE and TD meet. At this point S we have $SR' \cdot VR = ST' \cdot VQ = SP' \cdot VP$. So we find the point S which is the center of force. So $PA/QB = VQ/VP$, and $QB/RC = VR/VQ$.

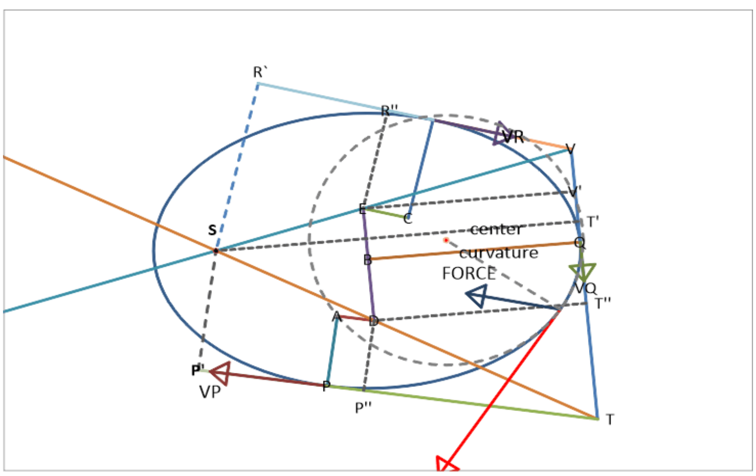

Figure 3. Newton's Method for Finding the Center of Force

2.1 Newton's Proof as Explained by Chandraseckar

Chandraseckar (1995) gave a proof based on Principia easier to understand. For the points *A*, *B*, *C* we note that $PA : QB : RC = VP^{(-1)}) : VQ^{(-1)} : VE^{(-1)}$ If S should be the (yet unspecified) center of attraction, drop the perpendiculars SP' and ST' to the tangents at *P* and *Q*. And draw (as illustrated) *CE*, *EBD*, and *DA* parallels to the tangents at *R*, *Q*, and *P* and intersecting at the points E and D. Now drop the perpendiculars *DP"* and *DT"* to the tangents at P and Q. Then by the $\frac{SP'}{ST'} = \frac{VQ}{VP} = \frac{AP}{BQ} = \frac{DP''}{DT''}$ or $\frac{SP'}{DP}'' = \frac{ST'}{DT}''$ So S, D, and T lie on the same line.

2.2 Finding the Law of the Force

Newton in principia gives us a mathematical method of finding the center of force for a body moving on an ellipse in Proposition V, Problem I. The same thinking can be applied also to the case of a hyperbola and also a parabola, only that in the last case the center of force is at infinite distance. The forces that studied Newton in the next pages of principia, for elliptical motion could be: a) An elastic force with center at the origin, b) a gravitational force with center at the left focus or c) a gravitational force with center at the right focus. In this article Newton's methods were extended to the case of hyperbola, where the forces could be a) an attractive gravitational force with center one pole of the hyperbola, b) a repulsive force with center the other pole of the hyperbola, c) an elastic force with center at the origin. It is possible to prove using the conservation of the angular momentum to prove that the force is proportional the inverse of the square of distance r from the focus. To find the law of force we use the law of areas in the form $r \cdot cos\omega \cdot V = h$ where ω is the angle between the line *r=SM* (see Figure 3) and the perpendicular on the ellipse. To find the cosω we use the parametric equations of the ellipse with semi-axes a and b as $x = a \cdot cos\varphi, y = b \cdot sin\varphi$ so $\frac{dy}{dx} = -\frac{a \cdot sin\varphi}{b \cdot cos\varphi}$, with unit vector $\vec{e} = \frac{a \cdot cos\varphi + c}{r}, \frac{b \cdot sin\varphi}{r}$ for perpendicular $\vec{k} = \frac{b \cdot cos\varphi}{\sqrt{\alpha^2 \cdot sin^{2\varphi} + b^2 \cdot cos^{2\varphi}}}, \frac{a \cdot sin\varphi}{\sqrt{a^2 \cdot sin^{2\varphi} + b^2 \cdot cos^{2\varphi}}}$ The Force that acts as centripetal: $F_k = F \cdot cos\omega = m \cdot \frac{V^2}{\rho} = \frac{m \cdot h^2}{(r \cdot cos\omega)^2 \cdot \rho}$ where ρ is the radius of curvature $\rho = \frac{\sqrt{(a^2 \cdot sin^{2\varphi} + b^2 \cdot cos^{2\varphi})^3}}{a \cdot b}$ so $F = \frac{m \cdot h^2}{r^2 \cdot cos3\omega \cdot \rho}$.

Physical entity	Center at left focus	Center at the center of ellipse	Center at the right focus
r	$a + c \cdot cos\varphi$	$\sqrt{\alpha^2 \cdot sin^{2\varphi} + b^2 \cdot cos^2\varphi}$	$a - c \cdot cos\varphi$
Cosω	$\frac{b \cdot cos\varphi}{\sqrt{\alpha^2 \cdot sin^{2\varphi} + b^2 \cdot cos^2\varphi}}$	$\frac{a + b \cdot cos\varphi}{r\sqrt{\alpha^2 \cdot sin^{2\varphi} + b^2 \cdot cos^2\varphi}}$	$\frac{b \cdot cos\varphi}{\sqrt{\alpha^2 \cdot sin^{2\varphi} + b^2 \cdot cos^2\varphi}}$
F	$\frac{m \cdot a \cdot h^2}{r^2 \cdot b^2}$	$\frac{m \cdot h^2 \cdot a \cdot b \cdot r}{(a+b)^3}$	$\frac{m \cdot a \cdot h^2}{r^2 \cdot b^2}$

By using the Excel file, the user can see that this method gives correctly the center of the force for either forces directed to the foci of the ellipse (gravitational attractive proportional to the reciprocal of the distance from the focus) or to the center of the ellipse (proportional to the distance of the particle from the center).

3. Finding the Center in the Case of Hyperbola

In the case of hyperbola there are three cases of force (We consider the movement on the right branch of the hyperbola). A) An attractive force with center of force the focus on the right side of the hyperbola. This force is reciprocal to the square of the distance from the focus. B) A repulsive force with center of force the left focus, which also is reciprocal to the square of the distance from the focus. C) A repulsive force with center of force the origin of the coordinates. In the figure the particle is moving to the left. It is clear that for attractive force the velocity is increasing as the particle moves to the left. For the case of elastic force the velocity increases fast as the body moves away to the right, while for repulsive force the velocity increases but there is a limit to the velocity.

We proceed as in the case of the ellipse. We find the points T and Z where the lines of the velocities $\overrightarrow{VP}$, $\overrightarrow{VQ}$ and $\overrightarrow{VQ}$, $\overrightarrow{VR}$ cut. We find as in ellipse the points, E and D and draw the lines TD and ZE and find the point S where the two lines cut each other. At the point S the law of areas holds for all points, so this is the center of force.

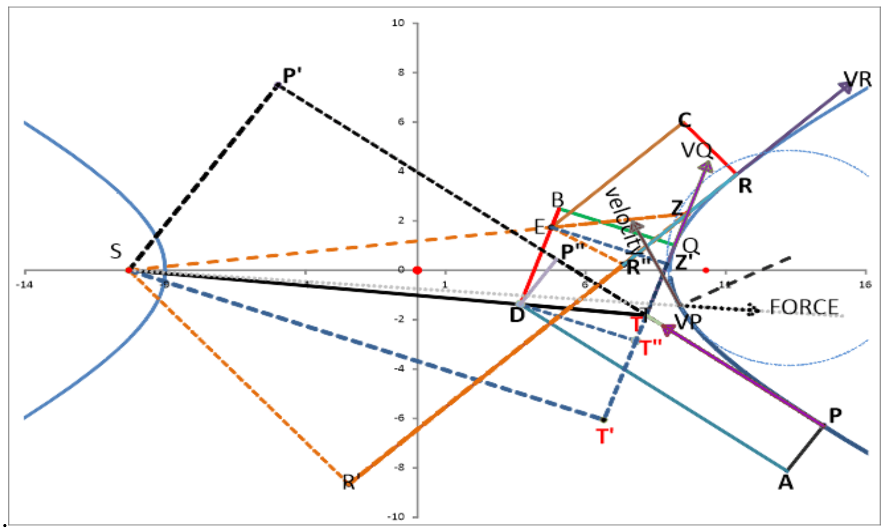

Figure 4. Application of Newton's Ideas for Hyperbola

3.1 Finding the Law of Force

To find the law of the force we need to calculate the Velocity and the radius of curvature ρ. The radius of curvature is $\rho = \frac{(a^2 \cdot sin(\varphi)^2 + b^2)^{\frac{3}{2}}}{a \cdot b \cdot cos(\varphi)^3}$ where φ is the angle from which we calculate the point of the curve: $(x, y) = \left(\frac{a}{cos(\varphi)}, b * \tan(\varphi)\right)$. The tangent is: $\frac{y}{dx} = \frac{b}{a * sin\varphi}$, and the unit vector on $\vec{r}$ is

$\vec{e} = \left(\frac{r_x}{r}, \frac{r_y}{r}\right)$. The perpendicular is found from $\vec{k} = \left(\frac{b}{\sqrt{b^2+\alpha^2\cdot\sin^2\varphi}}, \frac{-a\cdot sin\varphi}{\sqrt{b^2+\alpha^2\cdot\sin^2\varphi}}\right)$ and the angle ω between $\vec{e}$ and $\vec{k}$ is found from $cos\omega = \vec{k}\cdot\vec{e} = \frac{r_x\cdot b - r_y\cdot a\cdot sin\varphi}{r\cdot\sqrt{b^2+\alpha^2\cdot\sin^2\varphi}}$.

The velocity V is found from the law of areas: If $h = \vec{r}\times\vec{V}$ then $V = \frac{h}{r\cdot cos\omega}$ and we find the law of the force from the centripetal force $F_k = F\cdot cos\omega = m\cdot\frac{V^2}{\rho} = m\cdot\frac{h^2}{\rho\cdot(r\cdot cos\omega)^2}$ so $F = m\cdot\frac{h^2}{r^2\cdot\cos^3\omega\cdot\frac{(a^2\cdot sin(\varphi)^2+b^2)^{\frac{3}{2}}}{a\cdot b\cdot cos(\varphi)^3}}$.

	Center at left focus	Center at the center of ellipse	Center at the right focus
R	$\frac{a\cdot cos\varphi + c}{cos\varphi}$	$\sqrt{\alpha^2\cdot\cos^2\varphi + b^2\cdot\sin^2\varphi}$	$\frac{a\cdot cos\varphi - c}{cos\varphi}$
cosω	$\frac{b\cdot cos\varphi}{\sqrt{b^2+\alpha^2\cdot\sin^2\varphi}}$	$\frac{ab\cdot cos\varphi}{r\cdot\sqrt{b^2+\alpha^2\cdot\sin^2\varphi}}$	$\frac{b\cdot cos\varphi}{\sqrt{b^2+\alpha^2\cdot\sin^2\varphi}}$
F	$\frac{m\cdot h^2 a}{r^2\cdot b^2}$	$\frac{m\cdot h^2\cdot r}{(ab)^2}$	$\frac{m\cdot h^2\cdot a}{b^2\cdot r^2}$

4. Parabola

4.1 Constant Force

By applying the same method we get two parallel lines which show that the center of force is at an infinite distance. The user can change the initial speed and the angle of the initial velocity with the horizontal. As in the case of ellipse he can change the number of steps and see the construction of the figure. The lines TD and ZE are now parallels and so the center of force is at infinite distance.

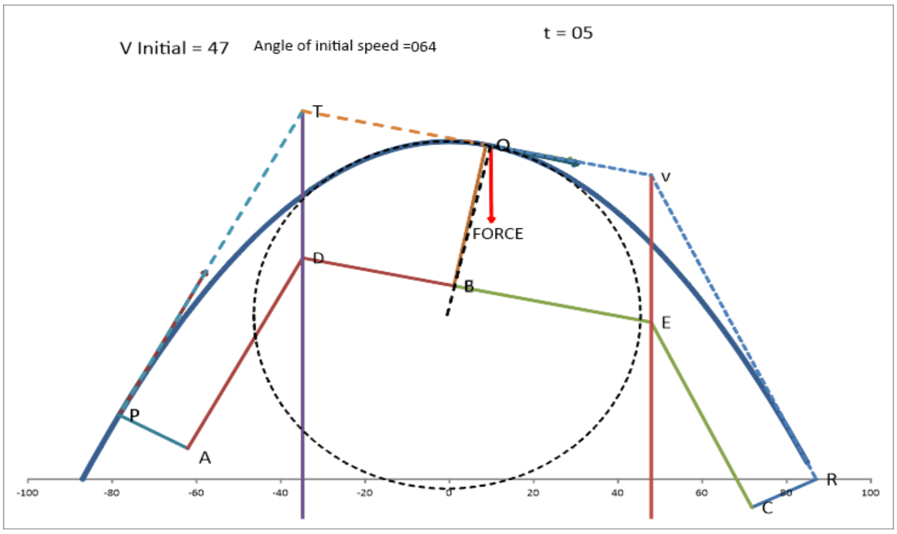

Figure 5. Application of Newton's Ideas for a Parabola for a Force Parallel to y Axis

We can prove that T and D have the same x_ component. The tangent at P is $i_p = \frac{dy}{dx} = \tan(\varphi) - \frac{j}{v_{0x}^2}(x_p - x_0)$ where $j = \frac{g}{v_{0x}^2}$ and φ the angle of the initial velocity v_0 with the horizontal. We find that the point T of the intersections of the lines of the velocities is given by: $x_T = \frac{S_{pq}}{i_p - i_q}$ where $S_{pq} - y_q - y_p + i_p x_p - i_q x_q$.

The points A, and B are having $x_A = x_p + d_p\, sin(\text{atan}(i_p))$ and $x_B = x_q + d_q\, sin(\text{atan}(i_q))$ where $d_p = \frac{k}{v_p}, d_q = \frac{k}{v_q}$ We find for the point D that it has as x – component $x_D = (S_{pq} - d_q(1 + i_q^2)^{0,5} + d_p(1 + t_p^2)^{0,5})/(i_p - i_q)$ but the last two terms on the numerator have a sum of zero since $-d_q(1 + i_q^2)^{0,5} + d_p(1 + t_p^2)^{0,5} = -\frac{k}{v_{0x}} + \frac{k}{v_{ox}} = 0$ since in the parabola the horizontal component is constant. For Ellipse and hyperbola the expressions become very complicated.

In the case of a force having a constant direction we can prove that this force is constant.

If the parabola is described by the equation y = a x ²then the radius of curvature is $\rho = \frac{\left|\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right|^3}{|d^2y/dx^2\ |} = \frac{\left|\sqrt{1+\tan^2\omega}\right|^3}{2a} = \frac{1}{2a\cdot\cos^3\omega}$ where ω is the angle of the tangent with the horizontal. Here the horizontal component of the velocity remains constant. So the velocity V is found from ω by V= V_0/cosω, where V_0 is the horizontal component of the velocity. The law for the force that acts on the body is found by:

$F\cdot cos\omega = m\cdot\frac{V^2}{\rho} = 2am\cdot V_0^2 cos\omega$ so F is a constant.

4.2 Force Directed to a Center

In this case the force is directed to the focus. The distance r of the point (x,y) on the parabola from the focus is found to be: $r = a\cdot x^2 + \frac{1}{4\cdot a}$ (Since $y = a\cdot x^2 + c, f = c + \frac{1}{4\cdot a}$). The angle θ between the r, and the perpendicular is found from $cos\theta = \left(\frac{x}{r}, \frac{y-f}{r}\right)\cdot\left(\frac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}, -\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right) = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}},$

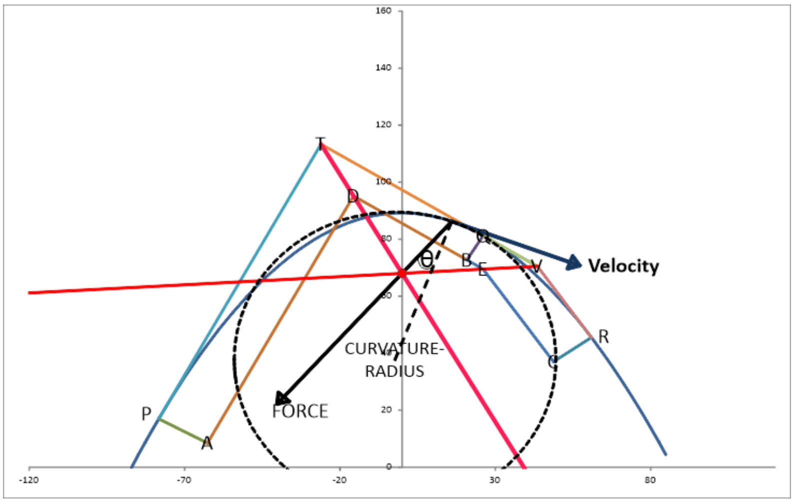

Figure 6. Parabolic Motion with a Force Directed towards the Focus and Proportional to 1/r²

Since this case the particle stars with zero velocity at infinity and gets its maximum velocity at the apex.

Since $\rho = \frac{\left|\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right|^3}{\left|\frac{d^2y}{dx^2}\right|} = \frac{1}{|\cos^3\theta \cdot 2a|}$ we see that $F_k = m \cdot \frac{V^2}{\rho \cdot cos\theta} = \frac{m \cdot V_O^2}{r^2 \cdot \rho \cdot \cos^3\theta} = \frac{2a \cdot m \cdot V_0^2}{r^2}$ and so $F\_k \propto \frac{1}{r^2}$. In this case if the particle starts with zero velocity at infinity and has maximum velocity at the apex.

5. Circular Movements

5.1 Circle and Forces which are Parallel

In this case a body describes a semicircle under the influence of a force which has a direction parallel to the y axis (Figure 7).

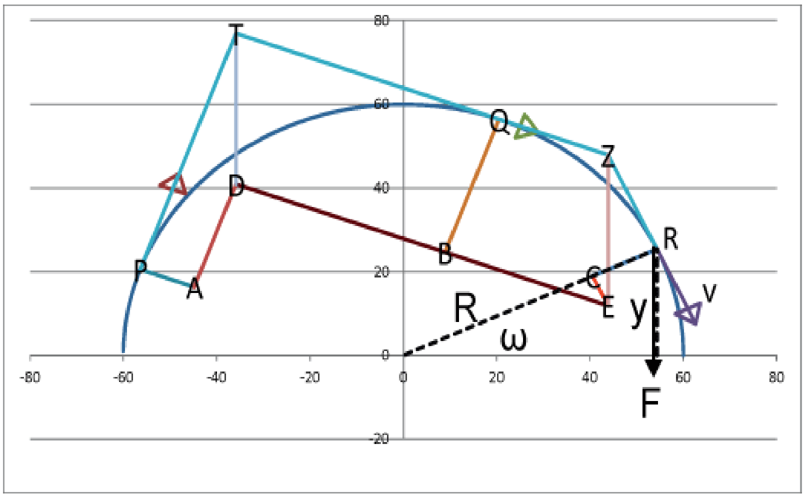

Figure 7. Application of Newton's Ideas for a Force Parallel to y-axis

In this case Newton proves that the force is inversely proportional to y^3. We can use the constancy of v_x to find the Centripetal Force. If v_0 is the velocity at the highest point then the horizontal component of the velocity v is $v_x = V_0 = V\cdot cos\omega$ where ω is the angle of the radius at the point P with the perpendicular. Then the centripetal force for the circular movement is $F \cdot cos\omega = \frac{m\cdot V^2}{R} = \frac{m\cdot V_0^2}{cos^2\omega\cdot R}$. Thus

$F \propto \frac{1}{cos^3\,\omega} \propto \frac{1}{y^3}$ since $y = R\ cos\omega$ Newton gives examples of circular movements as an introduction for the way to determine the centripetal attraction from the orbit. His examples can be also used to demonstrate that his method for finding the center of the force is applicable to these circular orbits.

A body describes a semicircle under the influence of a force which has a direction parallel to the y axis. In this case Newton proves that the force is inversely proportional to y^3. We can use the constancy of v_x to find the Centripetal Force. If v_0 is the velocity at the highest point then the horizontal component of the velocity v is $v_x=V_0=V*\sin\omega$ where ω is the angle of the radius at the point P with the horizontal. Then the centripetal force for the circular movement is $F \cdot sin\omega = \frac{mV^2}{R} = \frac{mV_0{}^2}{\sin^2\omega\cdot R}$ so F $\sim 1/\sin\omega^3$ $\sim 1/y^3$, since y =R sinω.

5.2 Attractive Force towards a Point on a Circumference

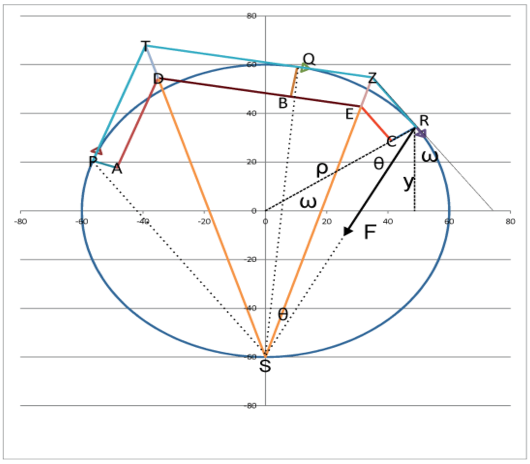

Figure 8. Application of Newton's Ideas for the Case of Force Directed to a Point on the Circumference

A circular (almost) orbit under the influence of a force directed to a point directed to a point on the circumference. In this case the force is proportional to the inverse of the 5th power of the distance *SR* of the body R from the center of force S. Here we apply the law of constant angular momentum: In the highest point V = V_0 so $m\cdot V_0\cdot 2\cdot\rho = mV\cdot r\cdot cos\theta$ where θ is the angle of the r = SR=2ρcosθ with the radius ρ at point R. The component of F o R is the centripetal Force so mV^2/ρ = F cosθ or

$mV_0^2/(2\rho*\cos^2\theta)= F*\cos\theta$ so $F \sim \frac{1}{cos\theta^3} \sim \frac{1}{SR^3}$.

5.3 Circular orbit with the Center of the Force inside the Circle

A circular orbit on a center which is inside the circle. In this case the force is expressed as $F = \frac{k}{RV^3 * SR^2}$ where SR is the distance of the center of force S from the body R and RV is the distance of the body P from the point V found by extending the line SR and finding the point V on the circle. Again here $v = V_0 \cdot \rho \cdot \frac{1+\alpha}{SP \cdot sin\varepsilon}$ If ε is the angle of the velocity at R with SR then $F \cdot sin\varepsilon = \frac{m \cdot V_0^2 (1+\alpha)^2}{(SR \cdot sin\varepsilon)^2}$ $sin\varepsilon = \frac{RV}{2\rho}$. By combining these we find that $F \sim \frac{1}{SR^2 * RV^\wedge 3}$.

6. Newton's Isogonal Spiral

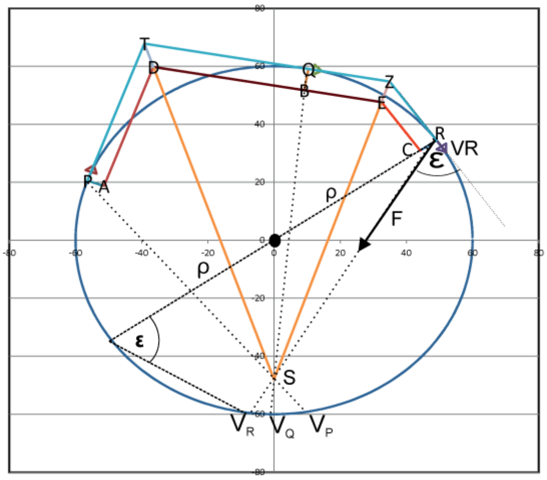

Figure 9. Application of Newton's Ideas for a Force Directed to a Center not on the Circumference

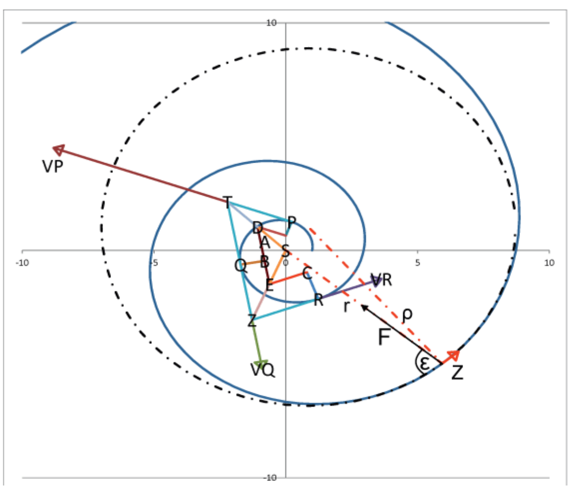

Figure 10. Application of Newton's Ideas for a Movement under the Influence of a Force Directed towards the Origin

The last application of the ideas is on Newton's isogonal spiral which is Proposition 9 Problem 3 *Let a body revolve in a spiral PQS intersection all its radii SP, SQ, … Of a given angle; it is required to find the law of the centripetal force tending toward the center of the spiral.* We can prove that the force is proportional to r^{-2} where r is the distance of the point to the point S. We apply again the law of areas and if at an initial point the velocity is V_0 and the initial radius of curvature is ρ_0 then at the point Z ρ ·V = ρ_0 ·V_0 To find the law of the force we note that the centripetal force will be F sinε =m ·V^2/ρ = m ·V_0^2 ρ_0 ?ρ^3. Since the radius of curvature is ρ = r/sinε and we find that $F{\sim}1/r3$.

7. Fixing the Velocities in Excel

To move the body around we need to have velocities that really correspond to the ellipse with semi-axes a,b and foci at c,-c: The velocity in the farthest point is given the value v_{oo} and then by the constancy of the angular momentum the velocity at the nearest point $v\_oo = voo \cdot \frac{a+c}{a-c}$ and then by the law of conservation of energy we can find the constant of the law by $\frac{1}{2} \cdot m \cdot voo^2 - \frac{k \cdot m}{a+c} = \frac{1}{2} m \cdot v_{oo}^2 - \frac{k \cdot m}{a-c}$.

We can proceed on the same manner for the case of hyperbola. In the case of hyperbola we can have for a particle moving on the right branch either an attractive gravitational force directed to the right focus, or a repulsive $-1/r^2$ or a repulsive directed to the center. We note that for the repulsive force from the left focus, we can see that the velocity for infinite distance is not zero but equal to $v_\infty = v\_00 \cdot (c + a)/b$ while for the elastic force the velocity is infinite.

8. Analytic Solution

The distance of the center of force (s_x, s_y) from the velocities is $d = \pm(sin\varphi s_x - cos\varphi s_y + cos\varphi y_1 - sin\varphi x_1)$ where *(x_1, y_1)* is the point where the velocity is V_1 and φ the angle of inclination. *For x_1 we take the negative since the numerator is negative (we expect s_y to be zero).*

By demanding that the law of areas is valid we get the equations: $d_1 V_1 = d_2 V_2 = d_3 V_3$ or: We denote by V_1y=V_1sinφ_1, V_1x=V_1cosφ_1, etc.

$$\left(V_{1y} + V_{2y}\right)S_x - (V_{1x} + V_{2X})S_Y = (\overrightarrow{rQ} X \overrightarrow{V_2} + \overrightarrow{rP}\ X\ \overrightarrow{V_1})$$
$$\left(V_{1y} + V_{3y}\right)S_x - (V_{1x} + V_{3x})S_Y = (\overrightarrow{rR} X\ \overrightarrow{V_3} + \overrightarrow{rP}\ X\ \overrightarrow{V_1})$$

By solving these equations we find the center.

Excel Files

For ellipses and parabolas, the _le is downloadable from:

http://www.kyriakosxolio.gr/2_dimensional_motiona/newtonstheoryofellipse2_2.xlsm

The corresponding _le for hyperbolas is downloadable from:

http://www.kyriakosxolio.gr/2_dimensional_motiona/HYPEROBOLAS_NEWTON_FIND_CENTER_without_macros.xlsm A zip _e with the excel _les for circular motions and Newton's spiral is downloadable from: http://www.kyriakosxolio.gr/2_dimensional_motiona/circular_spiral.zip

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