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Original Paper

A Prediction of Option Price via Two Volatility Computations with Application to a 50ETF Option

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Abstract

Financial derivative trading is integral to stock markets, leading to high option price volatility due to increased trading volume. Determining a reasonable option price is complex and requires extensive research in fields like Economics, Applied Mathematics, and Finance Engineering. The Black-Scholes (BS) equation provides a scientific pricing tool for options by considering five parameters: stock price (S), option strike price (K), risk-free interest rate (r), time to expiration (τ), and volatility (σ). Notably, all model parameters except volatility σ can be directly observed from market data, necessitating the determination of this parameter from historical data when applying the BS model in practice.

In this report, we examine two widely used computation methods for volatility. The first method involves a simple statistical calculation of historical data, resulting in the historical volatility (HV). The second method utilizes the BS model in a "backward" manner: given any previous option price and four other parameters, we solve the BS equation to obtain the implied volatility (IV). To determine such an IV parameter, we propose a generalized fixed-point iterative solver for solving a complex nonlinear equation. By employing a well-designed initial guess, we demonstrate that this fixed-point solver achieves global and rapid convergence.

The two volatilities lead to different predictions of the option price. This report examines if the BS model accurately predicts the option price using these volatilities. It presents an empirical study on a 50ETF option, where we use t-tests to check the hypothesis and determine the relationship between predicted and actual option prices. The study finds that using implied volatility in the BS model provides significantly more accurate predictions compared to historical volatility.

Keywords

Options pricing, Black–Scholes model, volatility, fixed point iteration, t-test, statistical computation

1. Introduction

During this competition, we mainly focus on how to price an option. For this reason, we designed two volatility computation methods, and applied the obtained volatility to the price prediction of several internationally renowned options such as Google. In Section 1.1, we introduce the concepts of an option, which is more abstract than a stock or a spot. Then, in Section 1.2, we introduce how to price an option, where we mainly focus on the Black-Scholes (BS) model which is a widely used finical tool. Finally, in Section 1.3, we briefly introduce the BS model and the main problem for using it in practice.

1.1 How to Price an Option

The primary goal of option pricing is to calculate the probability of an option being exercised at expiration and assign a price to it. Commonly used variables in mathematical models include the underlying asset price, exercise price, volatility, interest rate, and expiration date. These models also derive risk factors or sensitivities known as "Greeks" based on these inputs. The Greeks help traders determine how sensitive a trade is to price fluctuations, volatility changes, and time passage. Longer-dated options are more valuable because they have a higher probability of being profitable at expiration. Higher volatility and interest rates also lead to higher option prices. Marketable options require different pricing methods than non-marketable ones. Traded option prices may differ from predicted values but having a predicted value helps assess the probability of profiting from trading those options. In modern-day options market, Fischer Black and Myron Scholes' 1973 model (Black-Scholes formula) is commonly used for deriving theoretical prices for financial instruments with known expiration dates. However, there are other models available such as the Cox-Ross-Rubinstein binomial option pricing model and Monte Carlo simulations.

2. Basis of the Black-Scholes Model

BS model is a mathematical tool for pricing financial derivatives, such as options or warrants. It is proposed in 1970s by American economist, Myron Scholes and Fischer Black, and modified by Robert Merton. In this section, we briefly explain the math of this model and how to use it in practice.

2.1 Math of BS Model

Throughout this report, we consider a European call option (an extension to American option will be discussed in Section 6). Denote the strike price of an option by K *and* the maturity by T. We assume that there is no dividend payment and the base stock price S_t satisfies the geometric Brownian motion:

$$dS_t=\mu S_t dt+\sigma dW_t,$$

where W_t is the standard Brownian motion. Assume that there is a **$1** cash account at the initial time t=0, and this cash account has a value of exp(rt) at time t, where r is the risk-free interest rate. Therefore, it holds that

$$dB_t=rB_t dt.$$

Denote the price of call option at time t by $C(S,t)$. According to Itô's Lemma, we have

$$dC(S,t) = (\frac{\partial C}{\partial t} + \mu S_t \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2})dt + \sigma S_t \frac{\partial C}{\partial S} dW_t \ .$$

Next, we construct a self-financing portfolio (no external money inflows or outflows) Π_t. We assume that at the time t we hold x_t units of cash account and y_t units of stock. So, $\Pi_t=x_tB_t+y_tS_t$. We choose the value of x_t and y_t to replicate the value of the call option. Based on the self-financing assumption, we get

$d\prod_t=x_tdB_t+y_tdS_t=(rx_tB_t+\mu y_tS_t)dt+\sigma y_ts_tdW_t$.

By matching the corresponding terms of the previous two equations, it is clear that

$$y_t = \frac{\partial C}{\partial S}, rx_t B_t = \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} \ .$$

Let $V_0=C_0$ be the starting value of the self-financing portfolio, and then we have $V_t=C_t$ for any $t>0$. Substituting the first two equations into $C_t=x_tB_t+y_tS_t$ gives the following PDEs (the well-known BS equation):

$$\frac{\partial C}{\partial t} + rS_t \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} = rC \ . \qquad (2.1)$$

Now, we explain how to derive an analytic solution of the above BS equation

(2.1). First, this PDE can be transformed into the heat equation

$$\frac{\partial u}{\partial r} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} \ .$$

by employing a variable replacement

$\tau=T-t$, $u=Ce^{\tau r}$, $x=\ln S+(r-0.5\sigma)\tau$.

Let the boundary condition for the above heat equation be $u(x,0)=u_0(x)$. Then, we can represent the solution for the equation in closed form as

$$u(x,\tau) = \frac{1}{\sqrt{2\pi\tau\sigma}} \int_{-\infty}^{\infty} u_0(z) \exp(-\frac{(x-z)^2}{2\sigma^2\tau}) dz \ .$$

Correspondingly, for the BS equation (2.1) applied to the European call options with terminal condition $u_0(S_T, K)=\max\{S_t-K,0\}$, the solution of (2.1) is

$$u(S,\tau) = \int_{-d_2}^{\infty} (Se^{(r-\sigma^2/2)\tau+\sigma\sqrt{\tau}\varepsilon} - K) \frac{1}{\sqrt{2\pi}} \exp(-\varepsilon^2/2) d\varepsilon \ . \qquad (2.2)$$

It remains to the deal with the infinity integral in (2.2). To this end, we let $\varepsilon=(z-x)/(\sigma\sqrt{r})$. Then, when $z=\ln K$, it holds

$$\varepsilon = \frac{\ln(K/S) - (r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}} \ .$$

So, the solution in (2.2) is transformed to

$$u(S,\tau) = \int_{-d_2}^{\infty} (Se^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}\varepsilon} - K) \frac{1}{\sqrt{2\pi}} e^{-0.5\varepsilon^2} d\varepsilon \ .$$

By a tedious but very routine calculation and by noticing $C=re^{-\tau r}$, we have the following formula of the analytic solution of the BS equation (2.1) applied to European call options:

$$\begin{cases} C_\sigma = SN(d_1(\sigma)) - e^{-r\tau} SN(d_2(\sigma)) \\ d_1(\sigma) = \dfrac{\log(S/K) + (r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \\ d_2(\sigma) = \dfrac{\log(S/K) + (r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}} \end{cases} \quad (2.3)$$

where **N(·)** is the cumulative distribution function (CDF) of the normal distribution.

Similarly, the BS formula for pricing European put options is

$$P(S,t) = e^{-r(T-t)} KN(-d_2) - S_t N(-d_1).$$

where the definitions of d_1 and d_2 and **N(·)** are the same as in (2.3).

3. Compute the Volatility

As we emphasized in Section 2, identifying the parameter volatility σ is a crucial step for applying the BS model in practice. In this section, we introduce two methods for computing this parameter.

3.1 Historical Volatility (HV)

The historical volatility measures the dispersion of returns for an option or market index over a specific time period. It is typically calculated by finding the average deviation from the average price of a financial instrument during that time. Standard deviation is commonly used, but not the only method to calculate historical volatility. Higher historical volatility indicates higher risk in an option, but this can also present opportunities for both bullish and bearish outcomes.

3.2 Implied Volatility (IV)

An implied volatility (denoted by σ_{IV}) is a quantity that results in an option price which equals to the actual price after substituting into the BS model. With the known model parameters, i.e., the stock price (S), the strike price of the option (K), the risk-free interest rate (r), the time to expiration (τ), and the option price from the market (C_{Mar}), the parameter σ_{IV} is the solution of following nonlinear equation

$f(x)=C_{Mar}$, (3.2)

where the nonlinear function $f(x)$ is defined by

$$\begin{cases} f(x) = SN(d_1(x)) - e^{-r\tau} SN(d_2(x)) \\ d_1(x) = \dfrac{\log(S/K) + (r + x^2/2)\tau}{x\sqrt{\tau}} \\ d_2(x) = \dfrac{\log(S/K) + (r - x^2/2)\tau}{x\sqrt{\tau}} \end{cases}, \quad (3.3)$$

After solving (3.2), we have $\sigma_{IV}=x$. In (3.3), the function **N** is the cumulative distribution function of the normal distribution, that is

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} \exp(-z^2/2) dz .$$

In the following part, we introduce a generalized fixed-point (FP) iteration to handle (3.2).

With a non-zero free parameter η, the solution of this equation is identical to the following FP problem

$x=F(x)$, $F(x):=x+\eta[f(x)-C_{Mar}]$. (3.4)

A geometry explanation for the solution of the FP problem is as follows. Let $y=x$ and $y=F(x)$ be respectively the straight line and the curve be specified by the nonlinear function $F(x)$. The following is a toy example

$0.5=f(x)$, $f(x):=\log(4+x^{0.5})+x^2\sin(1+2x)$.

From the FP problem (3.4), it is natural to solve the original equation (3.2) via the following iteration.

$x_{l+1}=F(x_l)$, $l=0,1,...$, (3.5)

where l is the iteration index and x_0 is the initial guess. The convergence of the generated sequence $\{x_l\}$ depends on the initial guess and the involved parameter η. For example, for the above toy example, from the same initial guess $x_0=0.3$, the choice $\eta=0.25$ and $\eta=0.5$ results in very different result for the FP iteration (3.5).

Now, for the BS equation (3.2)-(3.3), we can solve the volatility, i.e., x, via the above FP iterations as well. To make a rapid convergence, the parameter η should be chosen as follows：

$$\eta = -\frac{x_l - x_{l-1}}{f(x_l) - f(x_{l-1})}, \quad (3.6)$$

i.e., the parameter η is determined dynamically during iterations. The FP iteration is

$$x_{l+1} = x_l - \frac{x_l - x_{l-1}}{f(x_l) - f(x_{l-1})}[f(x_l) - C_{Mar}], \quad (3.7)$$

where we need two starting values x_0 and x_1. In practice, starting from the first initial guess x_0 we generate x_1 by

$x_1=x_0-[f(x_0)-C_{Mar}]$, (3.8)

and by this idea we only need one starting value x_0. According to Li (2005), we use the following choice of x_0 which is based on an analytic expansion of the BS equation:

$$x_0 = \sqrt{\frac{2|log(S/K)|e^{r\tau}}{\tau}}. \qquad (3.9)$$

Numerically, we found that this initial guess the FP iteration converges rapidly.

3.3 Predict the Price of an Option

The price of an option can be predicted using the BS model in a "forward" model with either σ=σHV or σ=σIV for volatility. In practice, we follow these steps to make predictions: first, we obtain a σ based on today's data (S, K, r and τ)*. Then, when tomorrow's data becomes available, we calculate Cσ using (3.10) with the obtained σ and the new data. This procedure allows us to generate a series of estimated prices that can be compared to the actual market price through a t-test.

4. Data Information

We collect data for an option of 50ETF with number HO2308-C from June 19 to August 18 in 2023. HO2308-C is a call option started from June 19 and the expiration date is August 18 in 2023. All the data used here are collected from the website of China Financial Futures Exchange (CFFE) (http://www.cffex.com.cn).

5. Data Analysis and Interpretation

In this section, we analyze the collected data of option HO2308-C. We first calculate volatility using two methods: implied volatility (σIV) via the BS model and historical volatility (σHV) through statistical analysis. Using these volatilities, we predict the option price using the forward fashion of the BS model. Throughout this section, we assume a risk-free interest rate of r=0.025%. Specifically, for the first method, σIV is calculated using yesterday's market data (stock price S, option strike price K, and time to expiration τ), which is then used to predict today's option price. For the second method, σHV is calculated based on market data from the last 5 trading days according to equation (3.1). Finally, a t-test is conducted on predicted option prices for each strike and trading day obtained from both methods.

5.1 Predict the Option Price via BS+IV

With the data, we show in Figure 1 the computed implied volatility σ_{IV}; there are 21×42=882 values of σ_{IV}. Each value of σ_{IV} is obtained by solving the BS equation via the fixed-point (FP) iteration (cf. Section 3.2). For FP iteration, the tolerance is set to 10^{-6}. In Figure 2 and Figure 3, for each σ_{IV} we show the required FP iteration number and the residuals at the last FP iteration. From Figure 2 and Figure 3, we see that we need at most 8 FP iterations to arrive at the prescribed tolerance and this means our FP method is very efficient to handle the nonlinear BS equation.

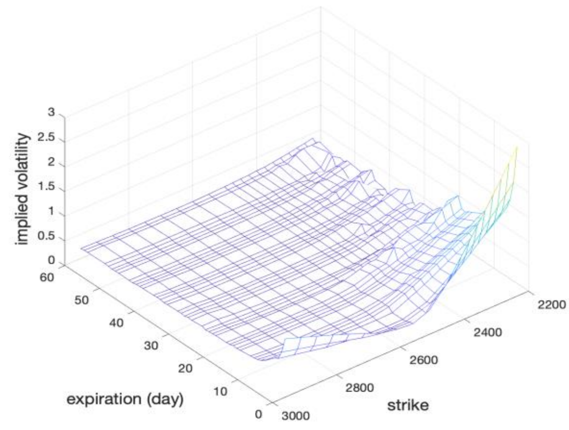

Figure 1. The Implied Volatility σ_{IV} Computed via the BS Model for the HO2308-C Option with Data Given in Table 4.1, Table 4.2, Table 4.3 and Table 4.4 in Section 4

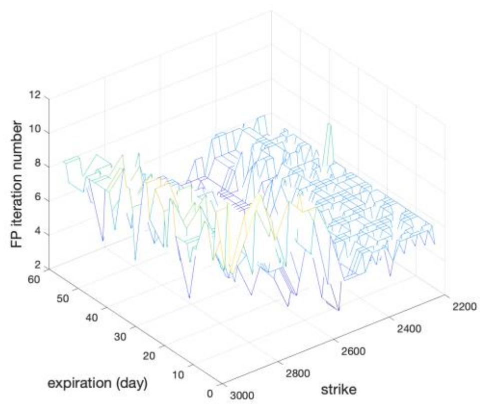

Figure 2. For Each Value of the Computed σ_{IV} in Figure 1, the Measured Iteration Number for the FP Method Introduced in Section 3.2

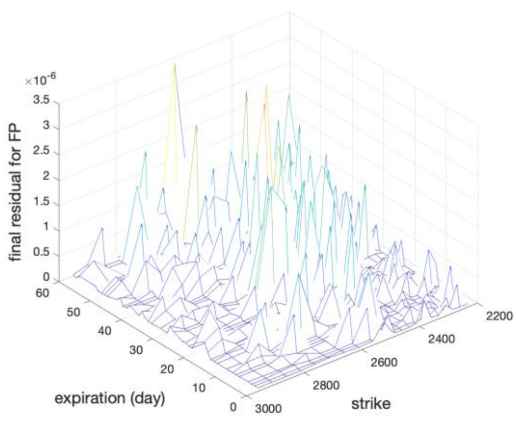

Figure 3. For Each Value of the Computed σ_{IV} in Figure 1, the Final Residual of the FP Iteration

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We now predict the option price by using the volatility σ_{IV}. To this end, we use the computed σ_{IV} on Jun19 to predict the price on the next day, i.e., Jun20. And then we move from Jun20 to Jun21, from Jun21 to Jun 26, from Jun26 to Jun27, and so on. We choose three representative trading days, Jun26, Jul21 and Aug7, for which we show in Figure 4 on the right the real option prices and the BS-predicted prices for each strike. The predicted price is obtained by using the implied volatility computed from the last trading day (left column in Figure 4), i.e., Jun 21, Jul20 and Aug4.

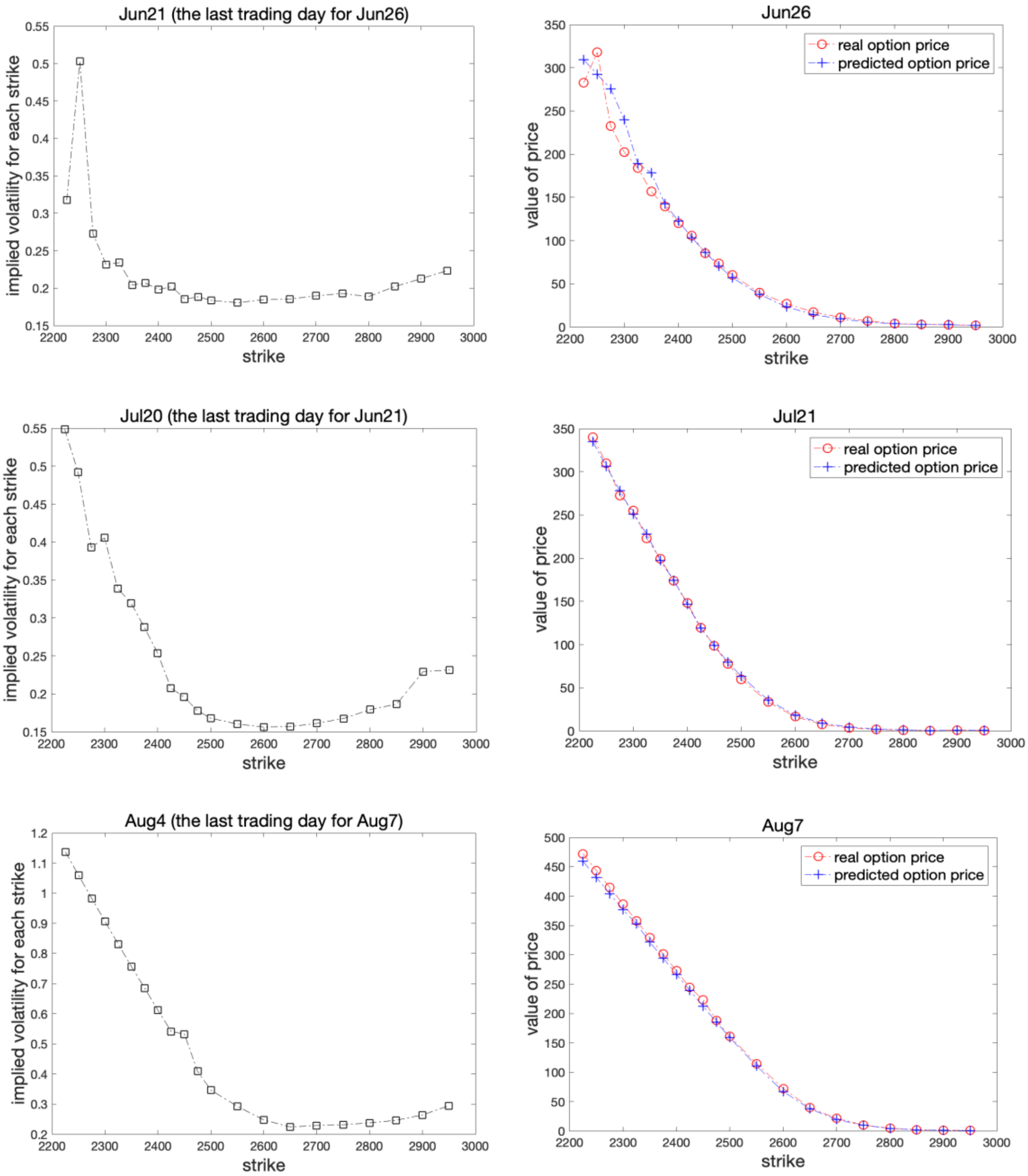
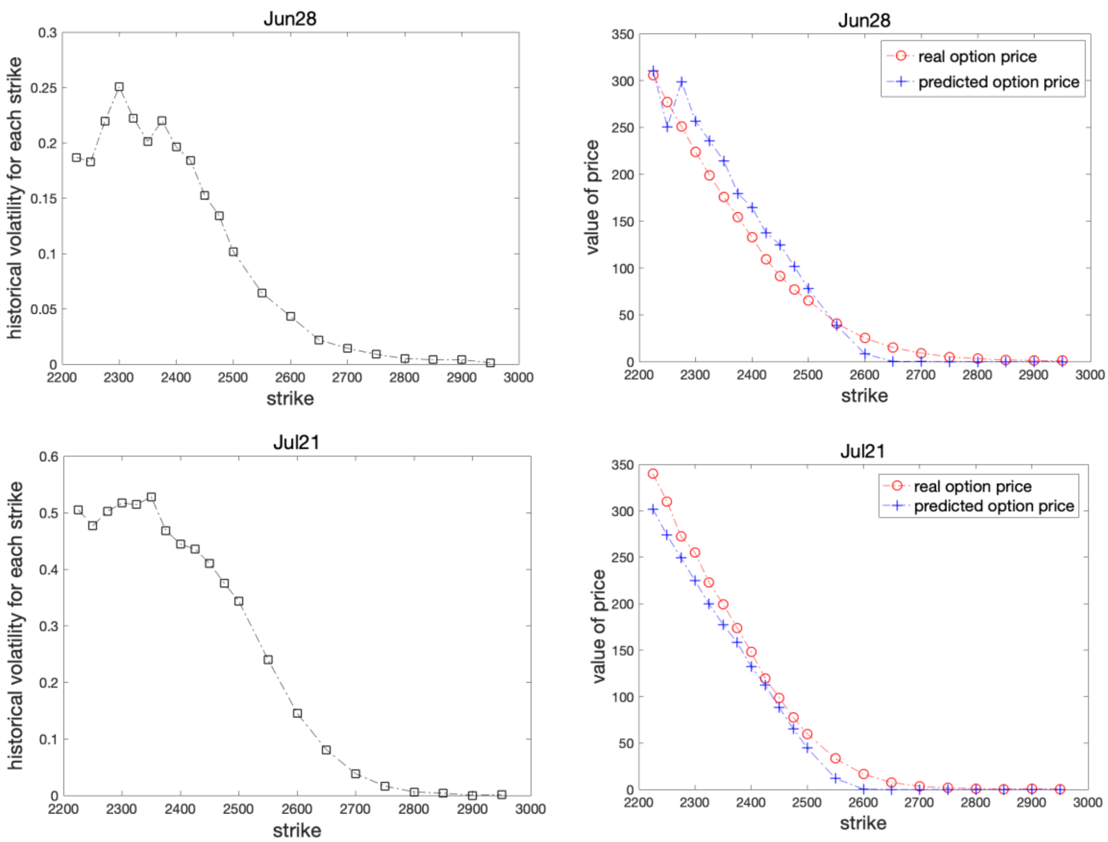


Figure 4. For Three Representative Trading Days, the Computed Implied Volatility σ_{IV} (left) and the Predicted Option Price by Using the Implied Volatility (right)

5.2 Predict the Option Price via BS+HV

We next consider another strategy to predict the option price: using the BS model together with the history volatility σ_{HV}. Such a σ_{HV} is computed via a statistical manner: we compute the volatility for the (M+1)-th trading day by using the real option prices of the last M trading days.

We compute the volatility (and then the option price) from Jun28 to Aug17. The option prices of the first five trading days, i.e., Jun19, Jun20, Jun21, Jun26 and Jun27, are used to predict the option price of Jun28. With these configurations, we show in Figure 5 the computed historical volatility (left) and the predicted option prices for each strike, where we consider three trading days, Jun28, Jul21 and Aug7. The BS model gives acceptable predictions of the real prices, even though the difference between the two prices is obviously visible.



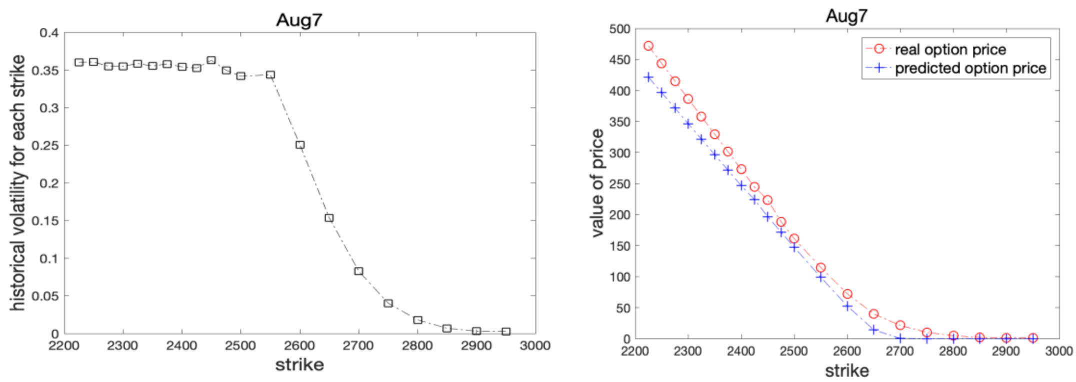


Figure 5. For Three Trading Days, the Computed Historical Volatility σ_{HV} (left) and the Predicted Option Price by Using the Implied Volatility (right)

5.3 Comparison and Interpretation

At the end of this section, we explain the predicted option prices in Section 5.1 and 5.2. For a given strike, we regard the predicted prices by using the implied volatility and the historical volatility by two samples as $\boldsymbol{C}_{IV}(K)$ and $\boldsymbol{C}_{HV}(K)$, which contain 42 price values for Jun19, Jun20,...., and Aug17. For $\boldsymbol{C}_{IV}(K)$, the first value (for Jun19) is set to be null and for $\boldsymbol{C}_{HV}(K)$ the first M values are set to be null; see illustration in the following Table 1. The prices observed from the market are denoted by $\boldsymbol{C}_{Mar}(K)$.

Table 1. Notations $C_{IV}(K)$ and $C_{HV}(K)$ Prepared for Two-sample t-test

	Jun19	Jun20	....	....	....	Aug17
$\boldsymbol{C}_{IV}(K)$	*null*	$C_{IV}(K,2)$	....	$C_{IV}(K,M+1)$	....	$C_{IV}(K,42)$
$\boldsymbol{C}_{HV}(K)$	*null*	....	*null*	$C_{HV}(K,M+1)$	....	$C_{HV}(K,42)$
$\boldsymbol{C}_{Mar}(K)$	$\boldsymbol{C}_{Mar}(K,1)$	$\boldsymbol{C}_{Mar}(K,2)$	....	$\boldsymbol{C}_{Mar}(K,M+1)$	....	$\boldsymbol{C}_{Mar}(K,42)$

We next make two hypotheses:

H0: $\boldsymbol{C}_{IV}(K)$ (or $\boldsymbol{C}_{HV}(K)$) and $\boldsymbol{C}_{Mar}(K)$ come from the same random distribution (with the same mean value and the same *unknown* variance), which means the difference between these two samples are not significant.

H1: $\boldsymbol{C}_{IV}(K)$ (or $\boldsymbol{C}_{HV}(K)$) and $\boldsymbol{C}_{Mar}(K)$ come from different random distribution, which means the difference between these two samples are significant.

Then, for each strike of the option HC2308-C we list the results of the two-sample t-test by using the MATLAB command t-test2.

In Table 2, we show the results for the two-sample t-test and we get two messages from these results. First, for $\boldsymbol{C}_{IV}(K)$ all the results are 0, which implies that for all the strikes the $\boldsymbol{C}_{IV}(K)$ and $\boldsymbol{C}_{Mar}(K)$ come from the same random distribution. This confirms a good coincidence of the predicted prices and the real prices in Figure 4 on the right. Second, for $\boldsymbol{C}_{HV}(K)$ (i.e., the option price predicted by using the historical volatility) the results are 1 for low and high strikes and are 0 for the middle strikes, which implies that for the low and high strikes $\boldsymbol{C}_{HV}(K)$ and $\boldsymbol{C}_{Mar}(K)$ come from different random distributions, while they come from the same distribution for the middle strikes. Again, this confirms the results in Figure 5 on the right: for the minimal and maximal strikes the difference between the predicted prices and the real prices is considerable, while the difference is relatively small for the middle strike K=2475.

Table 2. The Results for Two-sample t-test for $\boldsymbol{C}_{IV}(K)$ and $\boldsymbol{C}_{HV}(K)$

Strike	$\boldsymbol{C}_{IV}(K)$ and $\boldsymbol{C}_{Mar}(K)$	**Explanation**	$\boldsymbol{C}_{HV}(K)$ and $\boldsymbol{C}_{Mar}(K)$	**Explanation**
2225	0	Non-Significant	1	Significant
2250	0	Non-Significant	1	Significant
2275	0	Non-Significant	0	Non-Significant
2300	0	Non-Significant	0	Non-Significant
2325	0	Non-Significant	0	Non-Significant
2350	0	Non-Significant	0	Non-Significant
2375	0	Non-Significant	0	Non-Significant
2400	0	Non-Significant	0	Non-Significant
2425	0	Non-Significant	0	Non-Significant
2450	0	Non-Significant	0	Non-Significant
2475	0	Non-Significant	0	Non-Significant
2500	0	Non-Significant	0	Non-Significant
2550	0	Non-Significant	0	Non-Significant
2600	0	Non-Significant	0	Non-Significant
2650	0	Non-Significant	1	Significant
2700	0	Non-Significant	1	Significant
2750	0	Non-Significant	1	Significant
2800	0	Non-Significant	1	Significant
2850	0	Non-Significant	1	Significant
2900	0	Non-Significant	1	Significant
2950	0	Non-Significant	1	Significant

6. Conclusion and Further Work

The premium of an option, which is the price for buying or selling it, is crucial to understand when trading options. It depends on the probability of making a profit from buying or selling a stock at expiration. Buyers pay the premium while sellers receive it. An option gives the holder the right to buy or sell a specific amount of an underlying asset at a fixed price before its expiration date. Since it's a right and not an obligation, the holder can choose not to exercise it and let the option expire.

The price of an option is influenced by factors such as the current stock price, intrinsic value, time to expiration (or time value), volatility, interest rates, and cash dividends paid. Several option pricing models utilize these parameters to determine the fair market value of an option. Among them, the Black-Scholes (BS) model is widely recognized. Options are similar to other investments in that we need to understand what determines their price and use them effectively.

In this study, we examine the BS model's application in pricing European options and demonstrate that it can help investors anticipate overpricing and underpricing if all model constraints are thoroughly considered. Our findings indicate that as long as the volatility parameter is reasonably computed, the BS model provides acceptable predictions of real option prices based on market observations. Additionally, our research reveals that the price difference between observed and predicted values increases when stock movements deviate from investor expectations and volatility rises. This aligns with a similar outcome reported in Ali and Naima (2019), where authors compared call prices using different volatility estimations to assess pricing model efficiency.

To price an option using the BS model, we calculate volatility through two methods. The first method involves solving the BS equation for the implied volatility parameter using observed market data such as stock price, strike, expiration time, and risk-free interest rate. This is achieved by employing a generalized fixed-point algorithm with a novel choice of parameters and initial guess. Numerical experiments in Section 5 demonstrate rapid convergence of this proposed algorithm for all collected data. The second method entails calculating historical volatility by utilizing market's historical option prices.

We compare the two volatility parameters, σ_{IV} and σ_{HV}, we collected the real market data for an option of 50ETF with number HO2308-C from the website of China Financial Futures Exchange (CFFE) (http://www.cffex.com.cn). HO2308-C is a call option started from June 19 in 2023, and the expiration date is August 18 in 2023. It turns out that the volatility σ_{IV} computed by solving the BS equation provides much better precision for the real option price than the historical volatility σ_{HV}. The two-sample t-test supports this conclusion: it is shown that there is a significant difference (with high probability) between the estimated price using σ_{HV} and the real price, while such difference is not significant (with high probability) for σ_{IV}.

Based on this study (particularly the two computation methods for the volatility), our further research includes (but not limited to) the following two aspects. First, it would be interesting to generalize this study to put option by using the related BS equation $P(S,t)=e^{-r(T-t)}K\mathbf{N}(-d_2)-S_t\mathbf{N}(-d_1)$ (with d_1 and d_2 given by (2.3)). For put option, the computation of the historical volatility σ_{HV} is the same as that of the call option, but for the implied volatility σ_{IV} we have to change the parameter η for the fixed-point algorithm.

The second aspect is to extend the current study to American options, which can be exercised at any time before expiration. Pricing American options is more complex than European options, but it's more valuable as they play a significant role in today's financial market. The BS model is also useful for pricing American options, but its mathematical formula differs significantly from that of European options (see Mahato and Knowles (2020) for details). To use the BS model for pricing American options, we need to fix the volatility parameter. Our preliminary research shows that implied volatility provides better precision for real option prices observed in the market than historical volatility. However, obtaining implied volatility requires solving a corresponding nonlinear BS equation using a fixed-point algorithm that may converge slowly due to certain parameters like η.

References

Ali, B., & Naima, B. (2019). Options pricing by Monte Carlo simulation, Binomial Tree and BMS model: A comparative study. *Journal of Banking and Financial Economics*, *1*, 79-95. https://doi.org/10.7172/2353-6845.jbfe.2019.1.4

Bandi, F. M., & Perron, B. (2006). Long memory and the relation between implied and realized volatility. *J. Financ. Econom.*, *4*, 636-670. https://doi.org/10.1093/jjfinec/nbl003

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *J. Polit. Econ.*, 637-654. https://doi.org/10.1086/260062

Chicago Board Options Exchange: CBOE Volatility Indexes. 2016, March.

Latane, H. A., & Rendleman, R. J. (1976). Standard deviation of stock price ratios implied by option premia. *J. Finance*, *31*, 369-382. https://doi.org/10.1111/j.1540-6261.1976.tb01892.x

Li, S. (2005). A new formula for computing implied volatility. *Appl. Math. Comput.*, *170*, 611-625. https://doi.org/10.1016/j.amc.2004.12.034

Mahato, A., & Knowles, I. (2020). The inverse volatility problem for American options. *Discrete and Continuous Dynamical Systems -S*, *13*, 3473-3489.

Merton, R. C. (1973). Theory of rational option pricing. *Bell J. Econ. Manage. Sci.*, *4*, 141-183. https://doi.org/10.2307/3003143

Orlando, G., & Taglialatela, G. (2017). A review on implied volatility calculation. *J. Comput. Appl. Math.*, *320*, 202-220. https://doi.org/10.1016/j.cam.2017.02.002

Whaley, R. E. (1993). Derivatives on market volatility: Hedging tools long overdue. *J. Deriv.*, *1*, 71-84. https://doi.org/10.3905/jod.1993.407868