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Original Paper

Validity of Closed Ideals in Algebras of Series of Square Analytic Functions

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Abstract

We show the validity of a complete description of closed ideals of the algebra which is a commutative Banach algebra $\mathcal{A}_{\alpha_j^2}$*, that endowed with a pointwise operations act on Dirichlet space of algebra of series of analytic functions on the unit disk* $\mathbb{D}$ *satisfying the Lipscitz condition of order of square sequence* α_j^2 *obtained by (Brahim Bouya, 2008), we introduce and deal with approximation square functions which is an outer functions to produce and show results in* $\mathcal{A}_{\alpha_j^2}$*.*

Keywords

Dirichlet space, Lipschitz condition, Banach algebra, Besov algebras, Beurling-Rudin characterization, Beurling-Carleman-Domar resolvent method, F-property

1. Introduction

The Dirichlet space $\mathcal{D}$ consists of the sequence of square complex-valued analytic functions f_j^2 on the unit disk $\mathbb{D}$ with finite Dirichlet integral

$$\sum_j D\left(f_j^2\right) := \int_{\mathbb{D}} \sum_j \left|\left(f_j^2\right)'(z)\right|^2 dA(z) < +\infty,$$

where $dA(z) = \frac{1}{\pi}(1-\epsilon)d(1-\epsilon)dt^2$ denotes the normalized area measure on $\mathbb{D}$. Equipped with the pointwise algebraic operations and the series of norms

$$\sum_j \left\|f_j^2\right\|_{\mathcal{D}}^2 := \frac{1}{2\pi}\int_0^{2\pi} \sum_j \left|f_j^2\left(e^{it^2}\right)\right|^2 dt^2 + D\left(f_j^2\right) = \sum_{n=0}^{\infty}\sum_j (1+n)\left|\widehat{f_j^2}(n)\right|^2,$$

$\mathcal{D}$ becomes a Hilbert space. For $0 < \alpha_j^2 \le 1$, let $\mathrm{lip}_{\alpha_j^2}$ be the algebra of sequence of square analytic functions f_j^2 on $\mathbb{D}$ that are continuous on $\overline{\mathbb{D}}$ satisfing the Lipschitz condition of order α_j^2 on $\overline{\mathbb{D}}$:

$$\sum_j \left|f_j^2(z) - f_j^2(z-\epsilon)\right| = \sum_j o\left(|\epsilon|^{\alpha_j^2}\right) \qquad (|\epsilon| \to 0).$$

Note that this condition is equivalent to

$$\sum_j |(f_j^2)'(z)| = \sum_j o((1-|z|)^{\alpha_j^2-1}) \qquad (|z| \to 1^-).$$

Then, $lip_{\alpha_j^2}$ is a Banach algebra when equipped with series of norms

$$\sum_j \left\|f_j^2\right\|_{\alpha_j^2} := \sum_j \left\|f_j^2\right\|_\infty + \sup \sum_{\mathrm{j}} \{(1-|z|)^{1-\alpha_j^2} |(f_j^2)'(z)| : z \in \mathbb{D}\}.$$

Here $\sum_j \left\|f_j^2\right\|_\infty := \sup_{z\in\mathbb{D}} \sum_j |f_j^2(z)|$. Unlike as for the case when $0 < \alpha_j^2 \le \frac{1}{4}$, the inclusion $\mathrm{lip}_{\alpha_j^2} \subset \mathcal{D}$ always holds provided that $\frac{1}{4} < \alpha_j^2 \le 1$. In what follows, let $0 < \alpha_j^2 \le \frac{1}{4}$ and define $\mathcal{A}_{\alpha_j^2} := \mathcal{D} \cap \mathrm{lip}_{\alpha_j^2}$. It is easy to check that $\mathcal{A}_{\alpha_j^2}$ is a commutative Banach algebra when it is endowed with the pointwise algebraic operations and series of norms $\sum_j \left\|f_j^2\right\|_{\mathcal{A}_{\alpha_j^2}} := \sum_j \left\|f_j^2\right\|_{\alpha_j^2} + \sum_j D^{\frac{1}{2}}(f_j^2)$, $\quad (f_j^2 \in \mathcal{A}_{\alpha_j^2})$. In order to describe the closed ideals in subalgebras of the disc algebra $A(\mathbb{D})$, it is natural to make use of Nevanlinna's factorization theory. For $f_j^2 \in A(\mathbb{D})$ there is a canonical factorization $= C_{f_j^2} U_{f_j^2} O_{f_j^2}$, where $C_{f_j^2}$ is a constant, $U_{f_j^2}$ a sequence of square inner functions that is $\sum_j |U_{f_j^2}| = 1$ a.e on $\mathbb{T}$ and $O_{f_j^2}$ the sequence of square outer functions given by

$$\sum_j O_{f_j^2}(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \sum_j \frac{e^{i\theta^2}+z}{e^{i\theta^2}-z} \log\left|f_j^2(e^{i\theta^2})\right| d\theta^2\right\}.$$

Denote by $\mathcal{H}^\infty(\mathbb{D})$ the algebra of bounded analytic functions. Note that $\mathcal{A}_{\alpha_j^2}$ has the so-called F-property (Shirokov, 1988; Carleson, 1960): if $f_j^2 \in \mathcal{A}_{\alpha_j^2}$ and U is an inner function such that $f_j^2/U \in \mathcal{H}^\infty(\mathbb{D})$ then

$f_j^2/U \in \mathcal{A}_{\alpha_j^2}$ and $\sum_j \left\|f_j^2/U\right\|_{\mathcal{A}_{\alpha_j^2}} \le \sum_j C_{\alpha_j^2} \left\|f_j^2\right\|_{\mathcal{A}_{\alpha_j^2}}$, where $C_{\alpha_j^2}$ is independent of f_j^2. Korenblum (1972) has described the closed ideals of the algebra H_1^2 of sequence of square analytic functions f_j^2 such that $(f_j^2)' \in H^2$, where H^2 is the Hardy space. This result has been extended to some other Banach algebras of sequence of square analytic functions, by Matheson (1978) for $\mathrm{lip}_{\alpha_j^2}$ and by Shamoyan (1994) for the algebra $\lambda_{z-\epsilon}^{(n)}$ of sequence of square analytic functions f_j^2 on $\mathbb{D}$ such that $\sum_j |f_j^2)^{(n)}((z-2\epsilon)_1) - (f_j^2)^{(n)}((z-2\epsilon)_1 - \epsilon)| = o(\omega(|\epsilon|))$ as $|\epsilon| \to 0$, where n is a non negative integer and ω an arbitrary nonnegative non decreasing subadditive function on $(0, +\infty)$. Shirokov (1982, 1988) had given a complete description of closed ideals for Besov algebras $AB_{1+\epsilon,1+\epsilon}^{\left(\frac{1}{2}+\epsilon\right)}$ of sequence of square analytic functions and particularly for the case $\epsilon > 0$.

$$AB_{2,2}^{\left(\frac{1}{2}+\epsilon\right)} = \left\{(f_j^2 \in A(\mathbb{D}): \sum_{n\geq 0}\sum_{j}\left|\widehat{f_j^2}(n)\right|^2 (1+n)^{(1+2\epsilon)} < \infty\right\}.$$

Note that the case of $AB_{2,2}^{\frac{1}{2}} = A(\mathbb{D}) \cap \mathcal{D}$ the problem of description of closed ideals appears to be much more difficult (see Hedenmalm & Shields, 1990; El-Fallah, Kellay, & Ransford, 2006). Brahim Bouya (2008) described the structure of the closed ideals of the Banach algebras $\mathcal{A}_{\alpha_j^2}$. More precisely he proved that these ideals are standard in the sense of the Beurling-Rudin characterization of the closed ideals in the disc algebra (Hoffman, 1988), we show the general validation following (Brahim Bouya, 2008):

Theorem (1.1): If I is closed ideal of $\mathcal{A}_{\alpha_j^2}$, then

$$\mathfrak{T} = \left\{f_j^2 \in \mathcal{A}_{\alpha_j^2}: (f_j^2)_{\setminus E_{\mathfrak{T}}} = 0 \text{ and } f_j^2/U_{\mathfrak{T}} \in \mathcal{H}^\infty(\mathbb{D})\right\},$$

where $E_{\mathfrak{T}} := \left\{z \in \mathbb{T} : \sum_j f_j^2(z) = 0, \forall f_j^2 \in \mathfrak{T}\right\}$ and $U_{\mathfrak{T}}$ is the greatest common divisor of the inner parts of the non-zero functions in $\mathfrak{T}$.

Such characterization of closed ideals can be reduced further to a problem of approximation of outer functions using the Beurling– Carleman–Domar resolvent method. Define $d(\xi, E)$ to be the distance from $\xi \in T$ to the set $E \subset \mathbb{T}$. Suppose that $\mathfrak{T}$ is a closed ideal in $\mathcal{A}_{\alpha_j^2}$ such that $U_{\mathfrak{T}} = 1$. We have $Z_{\mathfrak{T}} = E_{\mathfrak{T}}$, where

$$Z_{\mathfrak{T}} := \left\{z \in \overline{\mathbb{D}}: \sum_j f_j^2(z) = 0, \qquad \forall f_j^2 \in \mathfrak{T}\right\}.$$

Next, for $f_j^2 \in \mathcal{A}_{\alpha_j^2}$ such that

$$\textstyle\sum_j |f_j^2(\xi)| \leq \sum_j Cd(\xi, E_{\mathfrak{T}})^{M_{\alpha_j^2}} \qquad (\xi \in \mathbb{T}),$$

where $M_{\alpha_j^2}$ is a positive constant depending only on $\mathcal{A}_{\alpha_j^2}$, we have $f_j^2 \in \mathfrak{T}$ (see section 3 for more precisions). Now, to show Theorem (1.1) we need Theorem (1.2) below, which states that every function in $\mathcal{A}_{\alpha_j^2} \setminus \{0\}$ can be approximated in $\mathcal{A}_{\alpha_j^2}$ by functions with boundary zeros of arbitrary high order.

Theorem (1.2): Let f_j^2 be a function in $\mathcal{A}_{\alpha_j^2} \setminus \{0\}$ and let $\epsilon \geq 0$. There exists a sequence of functions $\{(g_j)_n\}_{n=1}^{\infty} \subset A(\mathbb{D})$ such that

(i) For all $n \in \mathbb{N}$, we have $\sum_j (f_j^2)_n = \sum_j f_j^2 (g_j^2)_n \in \mathcal{A}_{\alpha_j^2}$ and $Lim_{n\to\infty} \sum_j \left\|(f_j^2)_n - f_j^2\right\|_{\mathcal{A}_{\alpha_j^2}} = 0$.

(ii) $\sum_j \left|(g_j^2)(\xi)\right| \leq \sum_j C_n d^{1+\epsilon}\left(\xi, E_{f_j^2}\right)$ $(\xi \in T)$,where $E_{f_j^2} := \{\xi \in T : \sum_j f_j^2(\xi) = 0\}$.

To show this Theorem, we give a refinement of the classical Korenblum approximation theory

(Korenblum, 1972; Matheson, 1978; Shamoyan, 1994; Shirokov, 1982; Shirokov, 1988).

2. Main Result on Approximation of Functions in $\mathcal{A}_{\alpha_j^2}$

Let $f_j^2 \in \mathcal{A}_{\alpha_j^2}$ and let $\{\gamma_n := (a_n, (a+\epsilon)_n)\}_{n\geq 0}$ be the countable collection of the (disjoint open) arcs of $\mathbb{T} \setminus E_{f_j^2}$. We can suppose that the arc lengths of γ_n are less than $\frac{1}{2}$. In what follows, we denote by Γ the union of a family of arcs γ_n. Define

$$\sum_j (f_j^2)_\Gamma (z) := \exp\left\{\frac{1}{2\pi}\int_\Gamma \sum_j \frac{e^{i\theta^2}+z}{e^{i\theta^2}-z}\log\left|f_j^2(e^{i\theta^2})\right|d\theta^2\right\}.$$

The difficult part in the proof of Theorem (1.2) is to establish the following

Theorem (2.1): Let $f_j^2 \in \mathcal{A}_{\alpha_j^2}\setminus\{0\}$ be an outer function such that $\sum_j \left\|f_j^2\right\|_{\mathcal{A}_{\alpha_j^2}} \leq 1$ and let $\epsilon \geq 1$ and $\epsilon > 0$. Then we have

$$f_j^{2(1+\epsilon)}\ (f_j)_\Gamma^{2(1+\epsilon)} \in \mathcal{A}_{\alpha_j^2} \text{ and } \sup_\Gamma \sum_j \left\|f_j^{2(1+\epsilon)}\ (f_j)_\Gamma^{2(1+\epsilon)}\ \right\|_{\mathcal{A}_{\alpha_j^2}} \leq C_{1+\epsilon,1+\epsilon}, \qquad (1)$$

where $C_{1+\epsilon,1+\epsilon}$ is a positive constant independent of Γ.

Remark (2.2): For a set $S \subset A(\mathbb{D})$, we denote by $co(S)$ the convex hull of S consisting of the intersection of all convex sets that contain S. Set $\Gamma_n = \cup_{\epsilon\geq 0}\gamma_{n+\epsilon}$ and let f_j^2 be as in the Theorem (2.1) It is clear that the sequence $(f_j^{2(1+\epsilon)}(f_j)_{\Gamma_n}^{2(1+\epsilon)})$ converges uniformly on compact subsets of $\mathbb{D}$ to $f_j^{2(1+\epsilon)}$.

We use (2.1) to deduce, by the Hilbertian structure of $\mathcal{D}$, that there is a sequence $(h_j^2)_n \in co(\{f_j^{2(1+\epsilon)}(f_j)_{\Gamma_{1+\epsilon}}^{2(1+\epsilon)}\}_{\epsilon=0}^\infty)$ converging to $f_j^{2(1+\epsilon)}$ in $\mathcal{D}$. Also, by (Matheson, 1978, section 4), we obtain that $(h_j^2)_n$ converges to $f_j^{2(1+\epsilon)}$ in $\mathrm{lip}_{\alpha_j^2}$, for sufficiently large $(1+\epsilon)$ (in fact, we can show that this result remains true for every $\epsilon \geq 0$). Therefore $\sum_j \left\|(h_j^2)_n - f_j^{2(1+\epsilon)}\ \right\|_{\mathcal{A}_{\alpha_j^2}} \to 0$, as $n \to \infty$.

Define $\mathcal{J}(F)$ to be the closed ideal of all functions in $\mathcal{A}_{\alpha_j^2}$ that vanish on $F \subset \overline{\mathbb{D}}$. In the proof of Theorem (1.2), we need the following classical lemma (see Brahim Bouya, 2008), see for instance (Matheson, 1978, Lemma 4) and (Korenblum, 1972, Lemma 24).

Lemma (2.3): Let $f_j^2 \in \mathcal{A}_{\alpha_j^2}$ and E' be a finite subset of $\mathbb{T}$ such that $\sum_j f_j^2|E' = 0. Let\ \epsilon \geq 0$ be given. For every $\varepsilon > 0$ there is an outer function F in $\mathcal{J}(E')$ such that

(i) $\sum_j \left\|Ff_j^2 - f_j^2\right\|_{\mathcal{A}_{\alpha_j^2}} \leq \varepsilon$,

(ii) $|F(\xi)| \leq Cd^{1+\epsilon}(\xi, E')$ $(\xi \in \mathbb{T})$.

Proof of Theorem (1.2): Now, we can deduce the proof of Theorem (1.2) by using Theorem (2.1) and Lemma (2.3) Indeed, let f_j^2 be a sequence of functions in $\mathcal{A}_{\alpha_j^2}\backslash\{0\}$ such that $\sum_j \|f_j^2\|_{\mathcal{A}_{\alpha_j^2}} \leq 1$ and let $\epsilon > 0$. For $\epsilon \geq 0$ we have

$$\sum_j \left(f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} - f_j^2\right)' = \sum_j \left(O_{f_j^2}^{\frac{1}{1+\epsilon}} - f_j^2\right)(f_j^2)' + \sum_j \tfrac{1}{1+\epsilon} U_{f_j^2} O_{f_j^2}^{\frac{1}{1+\epsilon}} O'_{f_j^2}.$$

The F-property of $\mathcal{A}_{\alpha_j^2}$ implies that $O_{f_j^2} \in \mathcal{A}_{\alpha_j^2}$. Then, there exists $\eta_0 \in \mathbb{N}$ such that

$$\sum_j \left\|f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} - f_j^2\right\|_{\mathcal{A}_{\alpha_j^2}} < \frac{\epsilon}{3} \qquad (\epsilon \geq 0).$$

Set $\Gamma_n = \cup_{1+\epsilon \geq n} \gamma_{1+\epsilon}$ and $\alpha_j^2 \leq 1$ for a given $\epsilon \geq 0$. By Remark (2.2) applied to $O_{f_j^2}$ (with $\epsilon =>$ 0), there is a sequence $k_{n,1+\epsilon} \in co\left(\left\{(f_j)_{\Gamma_{1+\epsilon}}^{1+\epsilon}\right\}_{\epsilon=0}^{\infty}\right)$ such that

$$\sum_j \left\|O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}} k_{n,1+\epsilon} - O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}}\right\|_{\mathcal{A}_{\alpha^2}} < \frac{1}{1+\epsilon} \quad (n \in \mathbb{N},\ \epsilon \geq 0).$$

It is clear that

$$\sum_j \left\|O_{f_j^2}^{\frac{1}{1+\epsilon}} (f_j)_{\Gamma_n}^{2(1+\epsilon)} - O_{f_j^2}^{\frac{1}{1+\epsilon}}\right\|_{\infty} \longrightarrow 0 \qquad (n \longrightarrow +\infty).$$

Then for every $\epsilon \geq 0$ we get

$$\sum_j \left\|O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{n,1+\epsilon} - O_{f_j^2}^{\frac{1}{1+\epsilon}}\right\|_{\infty} \longrightarrow 0 \qquad (n \longrightarrow +\infty).$$

So, there is a sequence $k_{1+\epsilon} \in co\left(\left\{(f_j)_{\Gamma_{1+\epsilon}}^{2(1+\epsilon)}\right\}_0^{\infty}\right)$ such that

$$\begin{cases} \displaystyle\sum_j \left\|O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}} k_{1+\epsilon} - O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}}\right\|_{\mathcal{A}_{\alpha_j^2}} \leq \frac{1}{1+\epsilon} & (\epsilon \geq 0), \\ \displaystyle\sum_j \left\|O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - O_{f_j^2}^{\frac{1}{1+\epsilon}}\right\|_{\infty} \leq \frac{1}{1+\epsilon} & (\epsilon \geq 0). \end{cases}$$

We have

$\sum_j \left(f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}}\right)' = \sum_j \left((f_j^2)' - U_{f_j^2} O'_{f_j^2}\right)\left(O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - O_{f_j^2}^{\frac{1}{1+\epsilon}}\right) + \sum_j \left(U_{f_j^2} O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}} k_{1+\epsilon} - O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}}\right)'$ Since $\sum_j \left\|O_{f_j^2}\right\|_{\mathcal{A}_{\alpha_j^2}} \leq \sum_j C_{\alpha_j^2} \|f_j^2\|_{\alpha_j^2} \leq \sum_j C_{\alpha_j^2}$, we obtain $\sum_j \left\|f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}}\right\|_{\mathcal{A}_{\alpha_j^2}} \sum_j \left\|f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}}\right\|_{\infty} +$

$sup_{z\in\mathbb{D}}\left\{\sum_j (1-|z|)^{1-\alpha_j^2} \left|\left(f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}}\right)'(z)\right|\right\} + \sum_j D^{\frac{1}{2}}\left(f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}}\right) \leq$

$\sum_j \left\|f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}}\right\|_{\infty} + \sum_j C_{\alpha_j^2} \|f_j^2\|_{\alpha_j^2} \left\|O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - O_{f_j^2}^{\frac{1}{1+\epsilon}}\right\|_{\infty} +$

$sup_{z\in\mathbb{D}}\left\{\sum_j(1-|z|)^{1-\alpha_j^2}\left|\left(O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}}k_{1+\epsilon}-O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}}\right)'(z)\right|\right\}+C\sum_j\left\|O_{f_j^2}^{\frac{1}{1+\epsilon}}k_{1+\epsilon}-O_{f_j^2}^{\frac{1}{1+\epsilon}}\right\|_\infty+\sum_j D^{\frac{1}{2}}(f_j^2)+$ $CD^{\frac{1}{2}}\sum_j\left(O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}}k_{1+\epsilon}-O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}}\right)\le\sum_j C_{\alpha_j^2}\left\|O_{f_j^2}^{\frac{1}{1+\epsilon}}k_{1+\epsilon}-O_{f_j^2}^{\frac{1}{1+\epsilon}}\right\|_\infty+C\sum_j\left\|O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}}k_{1+\epsilon}-O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}}\right\|_{\mathcal{A}_{\alpha_j^2}}\le\sum_j\frac{C_{\alpha_j^2}}{1+\epsilon}$

Then, fix $\epsilon\ge 0$ such that

$$\sum_j\left\|f_j^2O_{f_j^2}^{\frac{1}{1+\epsilon}}k_{1+\epsilon}-f_j^2O_{f_j^2}^{\frac{1}{1+\epsilon}}\right\|_{\mathcal{A}_{\alpha_j^2}}<\epsilon/3\qquad(\epsilon\ge 0).$$

We have $k_{1+\epsilon}=\sum_{i\le j_{1+\epsilon}}\sum_j c_i f_{\Gamma_i}^{2(1+\epsilon)}$, where $\sum_{i\le j_{1+\epsilon}}c_i=1$. Set $E'_{1+\epsilon}=\cup_{i\le j_{1+\epsilon}}\partial\gamma_i$. Using Lemma (2.3), we obtain an outer function $F_{1+\epsilon}\in\mathcal{J}(E'_{1+\epsilon})$ such that $|F_{1+\epsilon}(\zeta)|\le C_{1+\epsilon}d^{1+\epsilon}(\zeta,E'_{1+\epsilon})$ for $\zeta\in T$ and

$$\sum_j\left\|f_j^2O_{f_j^2}^{\frac{1}{1+\epsilon}}k_{1+\epsilon}F_{1+\epsilon}-f_j^2O_{f_j^2}^{\frac{1}{1+\epsilon}}k_{1+\epsilon}\right\|_{\mathcal{A}_{\alpha_j^2}}<\frac{1}{1+\epsilon},(\epsilon\ge 1).$$

Then fix $\epsilon\ \ge\ 0$ such that

$$\sum_j\left\|f_j^2O_{f_j^2}^{\frac{1}{1+\epsilon}}k_{1+\epsilon}F_{1+\epsilon}-f_j^2O_{f_j^2}^{\frac{1}{1+\epsilon}}k_{1+\epsilon}\right\|_{\mathcal{A}_{\alpha_j^2}}<\epsilon/3\quad(\epsilon\ge 0).$$

Consequently we obtain

$$\sum_j\left\|f_j^2O_{f_j^2}^{\frac{1}{1+\epsilon}}k_{1+\epsilon}F_{1+\epsilon}-f_j^2\right\|_{\mathcal{A}_{\alpha_j^2}}<\epsilon\qquad(\epsilon\ge 0).$$

It is not hard to see that

$$\sum_j\left|O_{f_j^2}^{\frac{1}{1+\epsilon}}k_{1+\epsilon}F_{1+\epsilon}(\xi)\right|\le\sum_j C_{1+\epsilon}d^{1+\epsilon}\left(\xi,E_{f_j^2}\right)\qquad(\xi\in\mathbb{T}).$$

Therefore $\sum_j(g_j^2)_{1+\epsilon}=\sum_j O_{f_j^2}^{\frac{1}{1+\epsilon}}k_{1+\epsilon}F_{1+\epsilon}$ is the desired series of sequence, which completes the proof of Theorem (1.2).

3. Beurling – Carleman – Domar Resolvent Methed

Since $\mathcal{A}_{\alpha_j^2}\subset\mathrm{lip}_{\alpha_j^2}$, then for all $f_j^2\in\mathcal{A}_{\alpha_j^2}$, $E_{f_j^2}$ satisfies the Carleson condition

$$\int_{\mathbb{T}}\sum_j\log\frac{1}{d(e^{it^2},E_{f_j^2})}dt^2<+\infty.$$

For $f_j^2\in\mathcal{A}_{\alpha_j^2}$, we denote by $B_{f_j^2}$ the Blashke product with zeros $Z_{f_j^2}\backslash E_{f_j^2}$, where $Z_{f_j^2}:=\{z\in\overline{\mathbb{D}}:\sum_j f_j^2(z)=0\}$. We begin with following lemma (see Brahim Bouya, 2008).

Lemma (3.1): Let $\mathfrak{I}$ be a closed ideal of $\mathcal{A}_{\alpha_j^2}$. Define $B_{\mathfrak{I}}$ to be the Blashke product with zeros $Z_{\mathfrak{I}}\backslash E_{\mathfrak{I}}$. There is a sequence of functions $f_j^2\in\mathfrak{I}$ such that $B_{f_j^2}=B_{\mathfrak{I}}$.

Proof. Let $g_j^2\in\mathfrak{I}$ and let B_n be the Blashke product with zeros $Z_{g_j^2}\cap\mathbb{D}_n$, where $\mathbb{D}_n:=\{z\in\mathbb{D}:|z|<\frac{n-1}{n},\ n\in\mathbb{N}\}$. Set $\sum_j(g_j^2)_n=\sum_j g_j^2/K_n$, where $K_n=B_n/I_n$ and I_n is the Blashke product with zeros $Z_{g_j^2} \cap \mathbb{D}_n$ .We have $(g_j^2)_n \in I$ for every n. Indeed, fix $n \in \mathbb{N}$.

It is permissible to assume that Z_{K_n} consists of a single point, say $Z_{K_n} = \{z-\epsilon\}$. Let $\pi : \mathcal{A}_{\alpha_j^2} \to \mathcal{A}_{\alpha_j^2}/\mathfrak{T}$ be the canonical quotient map. First suppose $(z-\epsilon) \notin Z_{\mathfrak{T}}$, then $\pi(K_n)$ is invertible in $\mathcal{A}_{\alpha_j^2}/\mathfrak{T}$. It follows that $\sum_j \pi(g_j^2)_n = \sum_j \pi(g_j^2)\pi^{-1}(K_n) = 0$, hence $(g_j^2)_n \in \mathfrak{T}$.

If $(z-\epsilon) \in Z_{\mathfrak{T}}$, we consider the following ideal $\mathcal{J}_{z-\epsilon} := \{f_j^2 \in \mathcal{A}_{\alpha_j^2} : f_j^2 I_n \in \mathfrak{T}\}$. It is clear that $\mathcal{J}_{z-\epsilon}$ is closed. Since $(z-\epsilon) \notin Z_{\mathcal{J}_{z-\epsilon}}$, it follows that K_n is invertible in the quotient algebra $\mathcal{A}_{\alpha_j^2}/\mathcal{J}_{z-\epsilon}$ and so $g_j^2/(I_n K_n) \in \mathcal{J}_{z-\epsilon}$. Hence $(g_j^2)_n \in \mathfrak{T}$. It is clear that $(g_j^2)_n$ converges uniformly on compact subsets of $\mathbb{D}$ to $\sum_j f_j^2 = \sum_J (g_j^2/B_{g_j^2})B_{\mathfrak{T}}$ and we have $\sum_J B_{f_j^2} = B_{\mathfrak{T}}$. In the sequel we prove that $f_j^2 \in \mathfrak{T}$. If we obtain

$$\sum_j \left| \left((g_j^2)_n \right)' (z) \right| \leq \sum_j o\left(\frac{1}{\epsilon^{1-\alpha_j^2}} \right) \qquad (z \in \mathbb{D}),$$

uniformly with respect to n, we can deduce by using (Matheson, 1978, Lemma 1) that

$\lim_{n\to+\infty} \sum_j \left\| (g_j^2)_n - f_j^2 \right\|_{\alpha_j^2} = 0$. Indeed, by the Cauchy integral formula

$$\sum_j \left((g_j^2)_n \right)' (z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \sum_j \frac{g_j^2(z-2\epsilon)\overline{K_n(z-2\epsilon)}}{4\epsilon^2} d(z-2$$

$$= \frac{1}{2\pi i} \int_{\mathbb{T}} \sum_j \frac{\left(g_j^2(z-2\epsilon) - g_j^2(z / |z|)\right)\overline{K_n(z-2\epsilon)}}{4\epsilon^2} d(z-2\epsilon) \qquad (z \in \mathbb{D}).$$

Then, for $z = (1-\epsilon)e^{i\theta^2} \in \mathbb{D}$

$$\sum_j \left((g_j^2)_n \right)' (z) \leq \frac{\|K_n\|_\infty}{2\pi} \int_{\mathbb{T}} \sum_j \frac{\left|g_j^2(z-2\epsilon) - g_j^2(z / |z|)\right|}{4|\epsilon|^2} |d(z-2\epsilon)|$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_j \frac{\left|g_j^2\left(e^{i(t^2+\theta^2)}\right) - g_j^2(e^{i\theta^2})\right|}{(2\epsilon-1)\cos t^2 + (1-\epsilon)^2} dt^2.$$

For all $\varepsilon > 0$, there is $\eta > 0$ such that if $|t^2| \leq \eta$, we have

$$\sum_j \left|g_j^2\left(e^{i(t^2+\theta^2)}\right) - g_j^2(e^{i\theta^2})\right| \leq \sum_j \varepsilon |t^2|^{\alpha_j^2} \qquad (\theta^2 \in [-\pi, +\pi]).$$

Then

$$\int_{-\pi}^{\pi}\sum_j \frac{\left|g_j^2\left(e^{i(t^2+\theta^2)}\right)-g_j^2(e^{i\theta^2})\right|}{(2\epsilon-1)\cos t^2+(1-\epsilon)^2}dt^2$$

$$\leq \varepsilon\int_{|t^2|\leq\eta}\sum_j \frac{|t^2|^{\alpha_j^2}}{\epsilon^2+4(1-\epsilon)t^2/\pi^2}dt^2$$

$$+\sum_j\|g_j^2\|_{\alpha_j^2}\int_{|t^2|\leq\eta}\sum_j \frac{|t^2|^{\alpha_j^2}}{\epsilon^2+4(1-\epsilon)t^2/\pi^2}dt^2$$

$$\leq \sum_j \frac{\varepsilon}{(1-\epsilon)^{\frac{1+\alpha_j^2}{2}}\epsilon^{1-\alpha_j^2}}\int_0^{+\infty}\sum_j\frac{u^{\alpha_j^2}}{1+(2u/\pi)^2}du$$

$$+\sum_j \frac{\|g_j^2\|_{\alpha_j^2}}{(1-\epsilon)^{\frac{1+\alpha_j^2}{2}}\epsilon^{1-\alpha_j^2}}\int_{|u|\geq\frac{\eta\sqrt{1-\epsilon}}{\epsilon}}\sum_j\frac{u^{\alpha_j^2}}{1+(2u/\pi)^2}du$$

$$\leq \sum_j \varepsilon O\left(\frac{1}{\epsilon^{1-\alpha_j^2}}\right)+\sum_j\|g_j^2\|_{\alpha_j^2}O\left(\frac{1}{\epsilon^{1-\alpha_j^2}}\right).$$

We obtain

$$\int_{-\pi}^{\pi}\sum_j \frac{\left|g_j^2\left(e^{i(t^2+\theta^2)}\right)-g_j^2(e^{i\theta^2})\right|}{(2\epsilon-1)\cos t^2+(1-\epsilon)^2}dt^2 \leq \sum_j\|g_j^2\|_{\alpha_j^2}O\left(\frac{1}{\epsilon^{1-\alpha_j^2}}\right). \tag{2}$$

Consequently

$$\sum_j\left|\left(\left(g_j^2\right)_n\right)'(z)\right| \leq \sum_j\|g_j^2\|_{\alpha_j^2}O\left(\frac{1}{\epsilon^{1-\alpha_j^2}}\right) \quad (z\in\mathbb{D}).$$

By the F-property of $\mathcal{A}_{\alpha_j^2}$, we have $\sum_j\left\|\left(g_j^2\right)_n\right\| \leq \sum_j C_{\alpha_j^2}\left\|\left(g_j^2\right)_n\right\|_{\mathcal{A}_{\alpha_j^2}}$. Using the Hilbertian structure of $\mathcal{D}$, we deduce that there is a sequence $(h_j^2)_n \in co(\{\left(g_j^2\right)_k\}_{k=n}^{\infty})$ converging to f_j^2 in $\mathcal{D}$. It is clear that $\left(h_j^2\right)_n \in \mathfrak{T}$ and $\lim_{n\to+\infty}\sum_j\left\|\left(h_j^2\right)_n - f_j^2\right\|_{\alpha_j^2} = 0$. Then $\lim_{n\to+\infty}\sum_j\left\|\left(h_j^2\right)_n - f_j^2\right\|_{\mathcal{A}_{\alpha_j^2}} = 0$. Thus $f_j^2\in\mathfrak{T}$. This completes the proof of the lemma.

We can see that $\sum_j\left\|\left(g_j^2\right)_n\right\|_{\alpha_j^2} O\left(\frac{1}{\epsilon^{1-\alpha_j^2}}\right) = \sum_j O\left(\frac{1}{\epsilon^{1-\alpha_j^2}}\right)$.

As a consequence of Theorem (1.2), we can show Theorem (1.1) and deduce that each closed ideal of $\mathcal{A}_{\alpha_j^2}$ is standard. For the sake of completeness, we sketch here the proof, (see Brahim Bouya, 2008).

Proof of Theorem (1.1): Define γ on $\mathbb{D}$ by $\gamma(z)=z$ and let $\pi:\mathcal{A}_{\alpha_j^2}\to\mathcal{A}_{\alpha_j^2}/\mathfrak{T}$ be the canonical quotient map. Also, let $f_j^2\in\mathcal{J}(E_{\mathfrak{T}})$ be such that $f_j^2/U_{\mathfrak{T}}\in\mathcal{H}^\infty(\mathbb{D})$ and $(f_j^2)_n$ be the sequence in Theorem (1.2) associated to f_j^2 with $\epsilon\geq 2$. More exactly, we have
$\sum_j(f_j^2)_n = \sum_j f_j^2(g_j^2)_n$, where $\sum_j|\left(g_j^2\right)_n(\xi)| \leq \sum_j d^3(\xi, E_{f_j^2}) \leq d^3(\xi, E_{\mathfrak{T}})$. Define

$$\sum_j L_\lambda\big(f_j^2\big)(z) := \begin{cases} \sum_j \dfrac{f_j^2(z) - f_j^2(\lambda)}{z-\lambda} & \text{if } z \neq \lambda, \\ \sum_j (f_j^2)'(\lambda) & \text{if } z = \lambda. \end{cases}$$

Then

$$\sum_j \pi\big(f_j^2\big)(\pi(\gamma)-\lambda)^{-1} = \sum_j f_j^2(\lambda)(\pi(\gamma)-\lambda)^{-1} + \sum_j \pi\Big(L_\lambda\big(f_j^2\big)\Big). \quad (3)$$

It is clear that $(\pi(\gamma)-\lambda)^{-1}$ is an analytic function on $\mathbb{C}\backslash Z_{\mathfrak{T}}$. Note that the multiplicity of the pole $z_0 \in Z_{\mathfrak{T}} \cap \mathbb{D}$ of $(\pi(\gamma)-\lambda)^{-1}$ is equal to the multiplicity of the zero z_0 of $U_{\mathfrak{T}}$. Since $U_{\mathfrak{T}}$ divides f_j^2, then according to (3) we can deduce that $\sum_j \pi\big(f_j^2\big)(\pi(\gamma)-\lambda)^{-1}$ is a series of square analytic functions on $\mathbb{C}\backslash E_{\mathfrak{T}}$. Let $|\lambda| > 1$, we have

$$\sum_j \big\|\pi\big(f_j^2\big)(\pi(\gamma)-\lambda)^{-1}\big\|_{\mathcal{A}_{\alpha_j^2}} \leq \sum_j \big\|f_j^2\big\|_{\mathcal{A}_{\alpha_j^2}} \sum_{n=0}^{\infty} \sum_j \|\gamma^n\|_{\mathcal{A}_{\alpha_j^2}} |\lambda|^{-n-1} \leq \sum_j \big\|f_j^2\big\|_{\mathcal{A}_{\alpha_j^2}} \frac{C}{(|\lambda|-1)^{\frac{3}{2}}}. \quad (4)$$

By Lemma (3.1), there is $g_j^2 \in \mathfrak{T}$ such that $B_{g_j^2} = B_{\mathfrak{T}}$. Let $k = \sum_j f_j^2\big(g_j^2/B_{g_j^2}\big)$. Then, $k = \sum_j (f_j^2/B_{\mathfrak{T}}) g_j^2 \in \mathfrak{T}$ and for $|\lambda| < 1$, we have $k(\lambda)(\pi(\gamma)-\lambda)^{-1} = -\pi\big(L_\lambda(k)\big)$.

Therefore

$$\sum_j \big\|\pi\big(f_j^2\big)(\pi(\gamma)-\lambda)^{-1}\big\|_{\mathcal{A}_{\alpha_j^2}} \leq \sum_j \big|f_j^2(\lambda)\big| \|(\pi(\gamma)-\lambda)^{-1}\|_{\mathcal{A}_{\alpha_j^2}} + \sum_j \big\|L_\lambda\big(f_j^2\big)\big\|_{\mathcal{A}_{\alpha_j^2}} \leq \sum_j \frac{\|L_\lambda(k)\|_{\mathcal{A}_{\alpha_j^2}}}{\Big|g_j^2/B_{g_j^2}\Big|(\lambda)} +$$

$$\sum_j \big\|L_\lambda\big(f_j^2\big)\big\|_{\mathcal{A}_{\alpha_j^2}} \leq \sum_j \frac{C(f_j^2,k)}{(1-|\lambda|)\Big|g_j^2/B_{g_j^2}\Big|(\lambda)} \leq \sum_j C\big(f_j^2,k\big) e^{\frac{C}{1-|\lambda|}} \quad (|\lambda| < 1). \quad (5)$$

We use (Taylor & Williams,1970, Lemmas 5.8 and 5.9) to deduce

$$\sum_j \big\|\pi\big(f_j^2\big)(\pi(\gamma)-\xi)^{-1}\big\| \leq \sum_j \frac{C\big(f_j^2,k\big)}{d(\xi,E_{\mathfrak{T}})^3} \qquad (1 \leq |\xi| \leq 2,\ \xi \notin E_{\mathfrak{T}}).$$

Then, we obtain $\xi \mapsto \sum_j |((g_j^2)_n)(\xi)| \big\|\pi(f_j^2)(\pi(\gamma)-\xi)^{-1}\big\| \in L^\infty(\mathbb{T})$.

With a simple calculation as in (Esterle, Strouse, & Zouakia, 1994, Lemma 2.4), we can deduce that

$$\sum_j \pi\big((f_j^2)_n\big) = \frac{1}{2\pi i}\int_{\mathbb{T}} \sum_j \big((g_j^2)_n\big)(\xi)(\pi(\gamma)-\xi)^{-1} d\xi\,.$$

Denote $\mathfrak{T}_{U_{\mathfrak{T}}}^{\infty}(E_{\mathfrak{T}}) := \Big\{h_j^2 \in A(\mathbb{D}) \colon (h_j^2)_{\backslash E_{\mathfrak{T}}} = 0 \text{ and } h_j^2 \,/\, U_{\mathfrak{T}} \in A(\mathbb{D})\Big\}$.

From (Hoffman, 1988, p. 81), we know that $\mathfrak{T}_{U_{\mathfrak{T}}}^{\infty}(E_{\mathfrak{T}})$ has an approximate identity $(e_{1+\epsilon})_{\epsilon \geq 0} \in \mathfrak{T}$ such that $\|e_{1+\epsilon}\|_\infty \leq 1$. $\mathfrak{T}$ is dense in $\mathfrak{T}_{U_{\mathfrak{T}}}^{\infty}(E_{\mathfrak{T}})$ with respect to the sup norm $\|\cdot\|_\infty$, so there exists $(u_{1+\epsilon})_{\epsilon \geq 0} \in \mathfrak{T}$ with $\|u_{1+\epsilon}\|_\infty \leq 1$ and $\lim_{1+\epsilon \to \infty} u_{1+\epsilon}(\xi) = 1$ for $\xi \in \mathbb{T}\backslash E_{\mathfrak{T}}$. Therefore $\sum_j \pi\big((f_j^2)_n\big) = \sum_j \pi\Big((f_j^2)_n - (f_j^2)_n u_{1+\epsilon}\Big) \to 0$ as $\epsilon \to \infty$. Then $(f_j^2)_n \in \mathfrak{T}$ and $f_j^2 \in \mathfrak{T}$.

Note that: if $\lim_{n\to\infty} \sum_j \big|(g_j^2)_n(\xi)\big| = \sum_j \big|(g_j^2)\big|\, |\xi|$ then, $\sum_j c d^{1+\epsilon}(\xi, E_{f_j^2}) = \sum_j d^3(\xi, E_{f_j^2})$.

4. Proof of Theorem (2.1)

The proof of Theorem (2.1) is based on a series of lemmas. In what follows, $C_{1+\epsilon}$ will denote a positive number that depends only on $1+\epsilon$, not necessarily the same at each occurrence. For an open subset Δ of $\mathbb{D}$, we put

$$\sum_j \left\|((h_j^2)'\right\|^2_{L^2(\Delta)} := \int_\Delta \sum_j \left|(f_j^2)'(z)\right|^2 dA(z).$$

We begin with the following key lemma (see Brahim Bouya, 2008).

Lemma (4.1): Let $f_j^2 \in \mathcal{A}_{f_j^2}$ be such that $\sum_j \left\|f_j^2\right\|_{\mathcal{A}_{\alpha_j^2}} \leq 1$ and let $\epsilon > 0$ be given. Then

$$\int_\gamma \sum_j \frac{\left|f_j^2(e^{it^2})\right|^{2(1+\epsilon)}}{d(e^{it^2})} dt^2 \leq \sum_j C_{1+\epsilon} \left\|(f_j^2)'\right\|^2_{L^2(\gamma)},$$

where $a, a+\epsilon \in E_{\mathfrak{X}}, \gamma = (a, a+\epsilon) \subset \mathbb{T} \backslash E_{f_j^2}$, $d(z) := \min\{|z-a|, |z-(a+\epsilon)|\}$ and $\Delta_\gamma := \{z \in D: z/|z| \in \gamma\}$.

Proof: Let $e^{it^2} \in \gamma$ and define $z_{t^2} := (1-d(e^{it^2}))e^{it^2}$. Since $|\gamma| < 1/2$, we obtain $|z_{t^2}| > \frac{1}{2}$. We have

$$\sum_j |f_j^2(e^{it^2})|^{2(1+\epsilon)} \leq \sum_j 2^{2\epsilon+1}\left(|f_j^2(e^{it^2}) - f_j^2(z_{t^2})|^{2(1+\epsilon)} + |f_j^2(z_{t^2})|^{2(1+\epsilon)}\right). \tag{6}$$

By Holder's inequality combined with the fact that $\sum_j \left\|f_j^2\right\|_\infty \leq \sum_j \left\|f_j^2\right\|_{\mathcal{A}_{\alpha_j^2}} \leq 1$, we get

$$\begin{aligned}\sum_j |f_j^2(e^{it^2}) - f_j^2(z_{t^2})|^{2(1+\epsilon)} &= \sum_j \left|f_j^2(e^{it^2}) - f_j^2(z_{t^2})\right|^{2\epsilon} \left|f_j^2(e^{it^2}) - f_j^2(z_{t^2})\right|^2 \\ &\leq 2^{2\epsilon}(1-|z_{t^2}|) \int_{|z_{t^2}|}^1 \sum_j \left|(f_j^2)'\left((1-\epsilon)e^{it^2}\right)\right|^2 (1-\epsilon) d(1-\epsilon) \\ &\leq 2^{2\epsilon+1} d\left(e^{it^2}\right) \int_0^1 \sum_j \left|(f_j^2)'\left((1-\epsilon)e^{it^2}\right)\right|^2 (1-\epsilon) d(1-\epsilon).\end{aligned}$$

Hence

$$\int_\gamma \sum_j \frac{\left|f_j^2\left(e^{it^2}\right) - f_j^2\left(z_{t^2}\right)\right|^{2(1+\epsilon)}}{d\left(e^{it^2}\right)} dt^2 \leq 2^{(2\epsilon+1)} \int_\gamma \int_0^1 \sum_j \left|(f_j^2)'\left(re^{it^2}\right)\right|^2 (1-\epsilon) d(1-\epsilon) dt^2 \leq$$

$$\sum_j 2^{(2\epsilon+1)} \pi \left\|(f_j^2)'\right\|^2_{L^2(\Delta_\gamma)}. \tag{7}$$

Since $d(e^{it^2}) \leq 1/2$, we obtain $\frac{d(e^{it^2})}{\sqrt{2}} \leq d(z_{t^2}) \leq \sqrt{2} d(e^{it^2})$. Put $d(z_{t^2}) = |z_{t^2} - \xi|$ and note that either $\xi = a$ or $\xi = a+\epsilon$. Let $z_{t^2}(u) = (1-u)z_{t^2} + u\xi \quad (0 \leq u \leq 1)$.

With a simple calculation, we can prove that for all $e^{it^2} \in \gamma$ and for all $u, 0 \leq u \leq 1$, we have $|z_{t^2}(u) - w| > \frac{1}{2}(1-u)d(e^{it^2})$ $(w \in \partial\Delta_\gamma)$, where $\partial\Delta_\gamma$ is the boundary of Δ_γ. Then $\mathbb{D}_{t^2,u} := \{z \in \mathbb{D}: |z - z_{t^2} t^2(u)| \leq \frac{1}{2}(1-u)d(e^{it^2})\} \subset \Delta_\gamma$, for all $e^{it^2} \in \gamma$ and for all $u, 0 \leq u \leq 1$. Since $\sum_j |(f_j^2)'(z)|$ is a series of subharmonic on $\mathbb{D}$, it follows that

$$\sum_j \left|(f_j^2)'(z_{t^2}(u))\right| \leq \frac{4}{\pi(1-u)^2 d^2(e^{it^2})} \int_{\mathbb{D}_{t,u}} \sum_j \left|(f_j^2)'(z)\right| dA(z)$$

$$\leq \frac{2}{\pi^{\frac{1}{2}}(1-u)\, d\,(e^{it^2})} \sum_j \left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)}.$$

Set $\varepsilon_{(1+\epsilon)} = 2\alpha_j^2 \epsilon$. We have

$$\sum_j \left|f_j^{2(1+\epsilon)}\,(z_{t^2})\right|^2 = \sum_j \left|f_j^{2(1+\epsilon)}(z_{t^2}) - f_j^{2(1+\epsilon)}(\xi)\right|^2$$

$$= (1+\epsilon)^2 |z_{t^2} - \xi|^2 \left| \int_0^1 \sum_j f_j^{2\epsilon}\big(z_{t^2}(u)\big)(f_j^2)'(z_{t^2}(u))du \right|^2$$

$$\leq C_{1+\epsilon} d^2\big(e^{it^2}\big) \left(\int_0^1 \sum_j |z_{t^2}(u) - \xi|^{\frac{\varepsilon_{1+\epsilon}}{2}} \left|(f_j^2)'\big(z_{t^2}(u)\big)\right| du \right)^2$$

$$\leq C_{1+\epsilon} d^{\varepsilon_{1+\epsilon}}\big(e^{it^2}\big) \left(\int_0^1 \frac{1}{(1-u)^{1-\frac{\varepsilon_{1+\epsilon}}{2}}} du \right)^2 \sum_j \left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)}^2$$

$$\leq C_{1+\epsilon} d^{\varepsilon_{1+\epsilon}}\big(e^{it^2}\big) \sum_j \left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)}^2.$$

Hence

$$\int_\gamma \sum_j \frac{\left|f_j^2(z_{t^2})\right|^{2(1+\epsilon)}}{d(e^{it^2})} dt^2 \leq \sum_j C_\rho \left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)}^2. \tag{8}$$

Therefore the result follows from (6), (7) and (8).

In the sequel, we denote by f_j^2 a series of square outer functions in $\mathcal{A}_{\alpha_j^2}$ such that $\sum_j \left\|f_j^2\right\|_{\mathcal{A}_{\alpha_j^2}} \leq 1$ and we fix a constant $1+\epsilon, 0 < \epsilon \leq 1$. By (Matheson, 1978 Theorem B), we have $f_j^{2(1+\epsilon)}\,(f_j)_\Gamma^{2(1+\epsilon)} \in \mathrm{lip}_{\alpha_j^2}$ and $\sum_j \left\| f_j^{2(1+\epsilon)}\,(f_j)_\Gamma^{2(1+\epsilon)} \right\|_{\mathrm{lip}_{\alpha_j^2}} \leq C_{1+\epsilon,1+\epsilon}$.

To prove Theorem (2.1) we need to estimate the integral $\int_{\mathbb{D}} \sum_j \left|f_j^{2(1+\epsilon)} (f_j^{2(1+\epsilon)})'\right|^2 dA(z)$. Define

$$\sum_j \big(f_j^2\big)_\Gamma(z) := \frac{1}{\pi} \int_\Gamma \sum_j \frac{e^{i\theta^2}}{(e^{i\theta^2}-z)^2} log\left|f_j^2(e^{i\theta^2})\right| d\theta^2. \tag{9}$$

Clearly we have $\sum_j (f_j^2)' = \sum_j f_j^2((g_j^2)_\Gamma + (g_j^2)_{\mathbb{T}\setminus\Gamma})$ and $\sum_j \left(\big(f_j\big)_\Gamma^{2(1+\epsilon)}\right)' = \sum_j (1+\epsilon)\big(f_j\big)_\Gamma^{2(1+\epsilon)}\,(g_j^2)_\Gamma$,

$$\sum_j f_j^{2(1+\epsilon)} (f_j^{2(1+\epsilon)})' = \sum_j (1+\epsilon) f_j^{2(1+\epsilon)} \big(f_j\big)_\Gamma^{2(1+\epsilon)}\,(g_j^2)_\Gamma \tag{10}$$

$$= \sum_j f_j^{2\epsilon}(1+\epsilon)(f_j^2)'\big(f_j\big)_\Gamma^{(1+\epsilon)} - \sum_j (1+\epsilon) f_j^{2(1+\epsilon)} \big(f_j\big)_\Gamma^{2(1+\epsilon)} (g_j^2)_{\mathbb{T}\setminus\Gamma}. \tag{11}$$

Since $\sum_j \|f_j^2\|_\infty \le 1$, it is obvious that $\sum_j \left\|(f_j)_\Gamma^{2(1+\epsilon)}\right\|_\infty \le 1$ and $\sum_j \|f_j^{2\epsilon}\|_\infty \le 1$. Hence, by (11) we get

$$\int_{\mathbb{D}} \sum_j \left|\left(f_j^{2(1+\epsilon)}\,(f_j)_\Gamma^{2(1+\epsilon)}\right)'\right|^2 dA(z) \le 2(1+\epsilon)^2 \int_{\mathbb{D}} \sum_j \left|\left(f_j^{2(1+\epsilon)}\,(f_j)_\Gamma^{2(1+\epsilon)}\right)'\right|^2 dA(z). \qquad (12)$$

We fix $\gamma = (a, a+\epsilon) \subset T \backslash E_{f_j^2}$ such that $\sum_j f_j^2(a) = \sum_j f_j^2(a+\epsilon) = 0$. Our purpose in what follows is to estimate the integral

$$\int_{\Delta_\gamma} \sum_j \left|\left(f_j^{2(1+\epsilon)}\,(f_j)_\Gamma^{2(1+\epsilon)}\right)'\right|^2 dA(z) \qquad (13)$$

which we can rewrite as

$$\int_{\Delta_\gamma} \sum_j \left|\left(f_j^{2(1+\epsilon)}\,(f_j)_\Gamma^{2(1+\epsilon)}\right)'\right|^2 dA(z) = \int_{\Delta_\gamma^1} + \int_{\Delta_\gamma^2},$$

Where

$$\Delta_\gamma^1 := \{z \in \Delta_\gamma : d(z) < 2(1-|z|)\}$$
$$\Delta_\gamma^2 := \{z \in \Delta_\gamma : d(z) \ge 2(1-|z|)\}.$$

The integral on the region Δ_γ^1. We begin with the following lemma (see Brahim Bouya, 2008).

Lemma (4.2):

$$\int_{\Delta_\gamma} \sum_j \frac{\left|f_j^2(z) - f_j^2(z/|z|)\right|^{2(1+\epsilon)}}{(1-|z|)^2} dA(z) \le \sum_j \frac{1}{2\alpha_j^2\,\epsilon} \left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)}.$$

Proof: Let $z = (1-\epsilon)e^{it^2} \in \Delta_\gamma$ and put $\varepsilon_{1+\epsilon} = 2\alpha_j^2\epsilon$. We have

$$\sum_j (1-\epsilon)\left|f_j^2\left((1-\epsilon)e^{it^2}\right) - f_j^2\left(e^{it^2}\right)\right|^{2(1+\epsilon)}$$
$$= \sum_j (1-\epsilon)\left|f_j^2\left((1-\epsilon)e^{it^2}\right) - f_j^2\left(e^{it^2}\right)\right|^{2\epsilon}\left|f_j^2\left((1-\epsilon)e^{it^2}\right) - f_j^2\left(e^{it^2}\right)\right|^2$$
$$\le (1-\epsilon)\epsilon^{1+\varepsilon_{(1+\epsilon)}} \int_{(1-\epsilon)}^1 \sum_j \left|(f_j^2)'((\tfrac{1}{2}+\epsilon)e^{it^2})\right|^2 d(\tfrac{1}{2}+\epsilon) \le (1$$
$$-\epsilon)\epsilon^{1+\varepsilon_{(1+\epsilon)}} \int_{(1-\epsilon)}^1 \sum_j \left|(f_j^2)'((\tfrac{1}{2}+\epsilon)e^{it^2})\right|^2 (\tfrac{1}{2}+\epsilon)\, d(\tfrac{1}{2}+\epsilon).$$

Therefore

$$\int_{\Delta_\gamma} \sum_j \frac{\left|f_j^2(z) - f_j^2(z/|z|)\right|^{2(1+\epsilon)}}{(1-|z|)^2} dA(z)$$
$$= \int_0^1 \left(\int_\gamma \sum_j \left|f_j^2\left((1-\epsilon)e^{it^2}\right) - f_j^2\left(e^{it^2}\right)\right|^{2(1+\epsilon)} \frac{(1-\epsilon)dt}{\pi}\right) \frac{d(1-\epsilon)}{\epsilon^2}$$
$$\le \sum_j \left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)} \int_{\mathbf{0}}^{\mathbf{1}} \frac{1}{\epsilon^{1-\varepsilon_{(1+\epsilon)}}} d(1-\epsilon).$$

This completes the proof.

Now, we can state the following result (see Brahim Bouya, 2008).

Lemma (4.3):

$$\int_{\Delta_\gamma^1} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} \left|\left((f_j^2)_\Gamma\right)'(z)\right|^2 dA(z) \le \sum_j C_{(1+\epsilon)} \|(f_j^2)'\|^2_{L^2(\Delta_\gamma)}.$$

Proof:. By Cauchy's estimate, it follows that $\sum_j |((f_j^2)_\Gamma)'((1-\epsilon)e^{it^2})| \le \frac{1}{\epsilon}$. Using Lemma (4.2), we get

$$\int_{\Delta_\gamma^1} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} \left|((f_j^2)_\Gamma)'(z)\right|^2 dA(z) \le \int_{\Delta_\gamma^1} \sum_j \frac{\left|f_j^2(z)\right|^{2(1+\epsilon)}}{(1-|z|)^2} dA(z) \le \sum_j C_{(1+\epsilon)} \left\|(f_j^2)'\right\|^2_{L^2(\Delta_\gamma)} +$$

$$2^{(2\epsilon+1)} \int_{\Delta_\gamma^1} \sum_j \frac{\left|f_j^2(z/|z|)\right|^{2(1+\epsilon)}}{(1-|z|)^2} dA(z). \tag{14}$$

Using Lemma (4.1), we obtain

$$\int_{\Delta_\gamma^1} \sum_j \frac{\left|f_j^2(z/|z|)\right|^{2(1+\epsilon)}}{(1-|z|)^2} dA(z) = \frac{1}{\mu} \int_{\Delta_\gamma^1} \sum_j \frac{\left|f_j^2\left(e^{it^2}\right)\right|^{2(1+\epsilon)}}{\epsilon^2} (1-\epsilon) d(1-\epsilon) dt^2 \le$$

$$\frac{C}{\pi} \int_\gamma \sum_j \frac{\left|f_j^2\left(e^{it^2}\right)\right|^{2(1+\epsilon)}}{\epsilon^2} dt^2 \le \sum_j C_{(1+\epsilon)} \left\|(f_j^2)'\right\|^2_{L^2(\Delta_\gamma)}. \tag{15}$$

The result of our lemma follows by combining the estimates (14) and (15).

The integral on the region Δ_γ^2 . In this subsection, we estimate the integral $\int_{\Delta_\gamma^2} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} \left|((f_j^2)_\Gamma)'(z)\right|^2 dA(z)$. Before this, we make some remarks. For $z \in \mathbb{D}$ define

$$a_\gamma(z) := \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_\Gamma \sum_j \frac{-\log\left|f_j^2(e^{it^2})\right|}{\left|e^{i\theta^2} - z\right|^2} d\theta^2 & if\ \gamma \nsubseteq \Gamma \\ \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{T}\setminus\Gamma} \sum_j \frac{-\log\left|f_j^2(e^{it^2})\right|}{\left|e^{i\theta^2} - z\right|^2} d\theta^2 & if\ \gamma \nsubseteq \Gamma. \end{cases}$$

Using the equation (10), it is easy to see that

$$\sum_j \left|f_j^2(z)^{1+\epsilon}((f_j^2)_\Gamma)'(z)\right|^2 \le 4 \sum_j \left|f_j^2(z)^{1+\epsilon} \frac{1}{2\pi} \int_\Gamma \frac{-\log\left|f_j^2(e^{it^2})\right|}{\left|e^{i\theta^2}-z\right|^2} d\theta^2\right|^2 . \tag{16}$$

Using the equation (11), it is clear that

$$\sum_j \left|f_j^2(z)^{1+\epsilon}((f_j^2)_\Gamma)'(z)\right|^2 \le 2 \sum_j \left|(f_j^2)'(z)\right|^2 + 8 \sum_j \left|f_j^2(z)^{1+\epsilon} \frac{1}{2\pi} \int_{\mathbb{T}\setminus\Gamma} \frac{-\log\left|f_j^2(e^{it^2})\right|}{\left|e^{i\theta^2}-z\right|^2} d\theta^2\right|^2 . \tag{17}$$

Then

$$\int_{\Delta_\gamma^2} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} \left|((f_j^2)_\Gamma)'(z)\right|^2 dA(z) \le 2 \sum_j \left\|(f_j^2)'\right\|^2_{L^2(\Delta_\gamma)} + 8 \int_{\Delta_\gamma^2} \sum_j f_j^2(z)^{2(1+\epsilon)} a_\gamma^2(z) dA(z). \tag{18}$$

Since $\log|f_j^2| \in L^1(\mathbb{T})$, we have

$$a_\gamma(z) \le \frac{C}{d^2(z)} \qquad \left(z \in \Delta_\gamma\right) \tag{19}$$

Given such inequality, it is not easy to estimate immediately the integral of the series of functions

$\sum_j |f_j^2(z)|^{2(1+\epsilon)} a_\gamma^2(z)$ on the whole Δ_γ^2. In what follows, we give a partition of Δ_γ^2 into three parts so that one can estimate the integral $\int \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} a_\gamma^2(z) dA(z)$ on each part. Let $z \in \Delta_\gamma^2$, three situations are possible :

$$a_\gamma(z) \leq 8\frac{|\log(d(z))|}{d(z)}, \tag{20}$$

$$8\frac{|\log(d(z))|}{d(z)} < a_\gamma(z) < 8\frac{|\log(d(z))|}{\epsilon} \tag{21}$$

$$8\frac{|\log(d(z))|}{\epsilon} \leq a_\gamma(z) \tag{22}$$

We can now divide Δ_γ^2 into the following three parts

$$\Delta_\gamma^{21} := \{z \in \Delta_\gamma^2 : z \text{ satisfying } (20)\},$$
$$\Delta_\gamma^{22} := \{z \in \Delta_\gamma^2 : z \text{ satisfying } (21)\},$$
$$\Delta_\gamma^{23} := \{z \in \Delta_\gamma^2 : z \text{ satisfying } (22)\},$$

The integral on the regions Δ_γ^{21} and Δ_γ^{23}. In this case we begin by the following (see Brahim Bouya, 2008).

Lemma (4.4):

$$\int_{\Delta_\gamma^{21}} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} a_\gamma^2(z) dA(z) \leq \sum_{\boldsymbol{j}} C_{(1+\epsilon)} \left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)}^2.$$

Proof: Using Lemma (4.2), we get

$$\begin{aligned}
&\int_{\Delta_\gamma^{21}} \sum_{\boldsymbol{j}} \left|f_j^2(z)\right|^{2(1+\epsilon)} a_\gamma^2(z) dA(z) \\
&\leq 2^{(1+\epsilon)} \int_{\Delta_\gamma^{21}} \sum_j \left|f_j^2(z)\right|^{\epsilon} \left|f_j^2(z) - f_j^2(z/|z|)\right|^{(\epsilon+2)} a_\gamma^2(z) dA(z) \\
&+ 2^{(1+\epsilon)} \int_{\Delta_\gamma^{21}} \sum_{\boldsymbol{j}} \left|f_j^2(z)\right|^{j} \left|f_j^2(z/|z|)\right|^{\epsilon+2} a_\gamma^2(z) dA(z) \\
&\leq C_{1+\epsilon} \int_{\Delta_\gamma} \sum_j \frac{\left|f_j^2(z) - f_j^2(z/|z|)\right|^{\epsilon+2}}{(1-|z|)^2} dA(z) \\
&+ C_{1+\epsilon} \int_{\Delta_\gamma^{21}} \sum_j \frac{\left|f_j^2(e^{it^2})\right|^{\epsilon+2}}{d^2(e^{it^2})} (1-\epsilon) d(1-\epsilon) dt^2 \\
&\leq \sum_j C_{1+\epsilon} \left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)}^2 + C_{1+\epsilon} \int_{\Delta_\gamma^{21}} \sum_j \frac{\left|f_j^2(e^{it^2})\right|^{\epsilon+2}}{d^2(e^{it^2})} d(1-\epsilon) dt^2 = I_{2,1}.
\end{aligned}$$

Let $e^{it^2} \in \gamma$ and denote by $(z-2\epsilon)_{t^2}$ the point of $\partial\Delta_\gamma^2 \cap \mathbb{D}$ such that $(z-2\epsilon)_{t^2}/|(z-2\epsilon)_{t^2}| = e^{it^2}$. We have

$$\left|e^{it^2} - (z-2\epsilon)_{t^2}\right| = 1 - |(z-2\epsilon)_{t^2}| = \frac{d((z-2\epsilon)_{t^2})}{2} \leq d(e^{it^2}).$$

Then

$$\int_{\Delta_\gamma^{21}}\sum_j\frac{\left|f_j^2\left(e^{it^2}\right)\right|^{\epsilon+2}}{d^2\left(e^{it^2}\right)}d(1-\epsilon)dt^2\le\int_{\Delta_\gamma^2}\sum_j\frac{\left|f_j^2\left(e^{it^2}\right)\right|^{\epsilon+2}}{d^2\left(e^{it^2}\right)}d(1-\epsilon)dt^2$$
$$=\int_\gamma\sum_j\frac{\left|f_j^2\left(e^{it^2}\right)\right|^{\epsilon+2}}{d^2\left(e^{it^2}\right)}\int_{\left|(z-2\epsilon)_{t^2}\right|}^1 d(1-\epsilon)dt^2\le\int_\gamma\sum_j\frac{\left|f_j^2\left(e^{it^2}\right)\right|^{\epsilon+2}}{d^2\left(e^{it^2}\right)}dt^2.$$

Using Lemma (4.1), we get $I_{2,1}\le\sum_j C_{1+\epsilon}\left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)}^2$. This proves the result.

Lemma (4.5):

$$\int_{\Delta_\gamma^{23}}\sum_j\left|f_j^2(z)\right|^{2(1+\epsilon)}a_\gamma^2(z)dA(z)\le CA\left(\Delta_\gamma\right),$$

where $A\left(\Delta_\gamma\right)$is the area measure of Δ_γ.

Proof: Set

$$\Lambda_\gamma:=\begin{cases}\Gamma & \text{for } \gamma\nsubseteq\Gamma,\\ \mathbb{T}\setminus\Gamma & \text{for } \gamma\subseteq\Gamma.\end{cases}$$

Let $z\in\Delta_\gamma^{23}$. We have

$$\sum_j\left|f_j^2(z)\right|=\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\sum_j\frac{2\epsilon-\epsilon^2}{\left|e^{i\theta^2}-z\right|^2}\log\left|f_j^2\left(e^{i\theta^2}\right)\right|d\theta^2\right\}$$

$$\le\exp\left\{\frac{1}{2\pi}\int_{\Lambda_\gamma}\sum_j\frac{2\epsilon-\epsilon^2}{\left|e^{i\theta^2}-z\right|^2}\log\left|f_j^2\left(e^{i\theta^2}\right)\right|d\theta^2\right\}=\exp\left\{-\epsilon a_\gamma(z)\right\}\le d^8(z).$$

Using (19), we obtain the result.

The integral on the region Δ_γ^{23}. Here, we will give an estimate of the following integral

$$\int_{\Delta_\gamma^{22}}\sum_j\left|f_j^2(z)\right|^{2(1+\epsilon)}a_\gamma^2(z)dA(z).$$

Before doing this, we begin with some lemmas (see Brahim Bouya, 2008).

The next one is essential for what follows. Note that a similar result is used by different authors: Korenblum (1972), Matheson (1978), Shamoyan (1994) and Shirokov (1982, 1988).

Lemma (4.6): Let $z\in\Delta_\gamma^{22}$ and let $\mu_z=1-\frac{8|\log(d(z))|}{a_\gamma(z)}$. Then

$$\sum_j\left|f_j^2(\mu_z z)\right|\le d^2(z). \tag{23}$$

Proof: Let $z\in\Delta_\gamma$ and let $\mu<1$. We have

$$\sum_j \left|f_j^2\,(\mu_z)\right| = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\sum_j \frac{1-(\mu(1-\epsilon))^2}{\left|e^{i\theta^2}-\mu z\right|^2}\log\left|f_j^2\,(e^{i\theta^2})\right|d\theta^2\right\}$$

$$\leq \exp\left\{\frac{1}{2\pi}\int_{\Lambda_\gamma}\sum_j \frac{1-(\mu(1-\epsilon))^2}{\left|e^{i\theta^2}-\mu z\right|^2}\log\left|f_j^2\,(e^{i\theta^2})\right|d\theta^2\right\}$$

$$= \exp\left\{-(1-\mu(1-\epsilon))\,\inf_{\theta^2\in\Lambda_\gamma}\left|\frac{e^{i\theta^2}-z}{e^{i\theta^2}-\mu z}\right|^2 a_\gamma(z)\right\}.$$

For $z \in \Delta_{\boldsymbol{\gamma}}^{\mathbf{22}}$ it is clear that $1-\mu z \leq d(z) \leq |e^{i\theta^2}-z|$ for all $e^{i\theta^2} \in \Lambda_\gamma$.

Then

$$\inf_{\theta^2\in\Lambda_\gamma}\left|\frac{e^{i\theta^2}-z}{e^{i\theta^2}-\mu z}\right|^2 \geq \frac{1}{2} \qquad \left(z \in \Delta_{\boldsymbol{\gamma}}^{\mathbf{22}}\right).$$

Thus

$$\sum_j \left|f_j^2\,(\mu_z z)\right| \leq \exp\left\{-\frac{1-\mu_z}{4}a_\gamma(z)\right\} \qquad \left(z \in \Delta_{\boldsymbol{\gamma}}^{\mathbf{22}}\right).$$

Then, we have

$\sum_j \left|f_j^2\,(\mu_z z)\right| \leq \exp\left\{-\frac{1}{4}(1-\mu_z)a_\gamma(z)\right\} = d^2(z) \quad \left(z \in \Delta_{\boldsymbol{\gamma}}^{\mathbf{22}}\right)$, which yields (23).

For $\epsilon > 0$ define $\gamma_{(1-\epsilon)} \coloneqq \{z \in \mathbb{D}\colon |z| = 1-\epsilon$ and $z/|z| \in \gamma\}$. Without loss of generality, we can suppose that $d(z) \leq \frac{1}{2}$, $z \in \Delta_{\boldsymbol{\gamma}}^{\mathbf{2}}$. We need the following (see Brahim Bouya, 2008).

Note that: we deduce that $\sum_j \left|f_j^2\,(\mu_z z)\right| \leq \frac{c'}{\left\|\log(\frac{1}{2})\right\|}$ where $c' = \frac{c}{16}$.

Lemma (4.7): Let $\epsilon > 0$. Then

$$\int_{\gamma_{(1-\epsilon)}\cap\Delta_{\boldsymbol{\gamma}}^{\mathbf{22}}}\sum_j \left|f_j^2\left((1-\epsilon)e^{it^2}\right) - f_j^2\left(\mu_{(1-\epsilon)e^{it^2}}(1-\epsilon)e^{it^2}\right)\right|^{2(1+\epsilon)} a_\gamma^2\left((1-\epsilon)e^{it^2}\right)(1-\epsilon)dt^2$$

$$\leq \sum_j \frac{C_{1+\epsilon}}{\epsilon^{1-\varepsilon_{(1+\epsilon)}}}\left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)}^2, \text{ where } \varepsilon_{(1+\epsilon)} = \alpha^2\epsilon.$$

Proof: Let $(1-\epsilon)e^{it^2} \in \Delta_{\boldsymbol{\gamma}}^{\mathbf{22}}$. Then

$$\sum_j \left|f_j^2\left((1-\epsilon)e^{it^2}\right) - f_j^2\left(\mu_{(1-\epsilon)e^{it^2}}(1-\epsilon)e^{it^2}\right)\right|^{\epsilon}\left[\left(1-\mu_{(1-\epsilon)e^{it^2}}\right)a_\gamma((1-\epsilon)e^{it^2})\right]^2$$

$$\leq 64\left(1-\mu_{(1-\epsilon)e^{it^2}}\right)^{\varepsilon_{(1+\epsilon)}}\log^2\left(d\left((1-\epsilon)e^{it^2}\right)\right) \leq C_{1+\epsilon}.$$

It is clear that $\epsilon \leq 1-\mu_{(1-\epsilon)e^{it^2}} \leq d\left((1-\epsilon)e^{it^2}\right) \leq \frac{1}{2}$ and so $\frac{1}{2} \leq d\left((1-\epsilon)e^{it^2}\right) \leq (1-\epsilon)$. We have

$$\int_{\gamma_{(1-\epsilon)}\cap\Delta_{\gamma}^{22}}\sum_{j}\left|f_j^2\left((1-\epsilon)e^{it^2}\right)-f_j^2\left(\mu_{(1-\epsilon)e^{it^2}}(1-\epsilon)e^{it^2}\right)\right|^{2(1+\epsilon)} a_\gamma^2\big((1-\epsilon)e^{it^2}\big)(1-\epsilon)dt^2$$

$$\leq C_{1+\epsilon}\int_{\gamma_{(1-\epsilon)}\cap\Delta_{\gamma}^{22}}\sum_{j}\frac{\left|f_j^2\left((1-\epsilon)e^{it^2}\right)-f_j^2\left(\mu_{(1-\epsilon)e^{it^2}}(1-\epsilon)e^{it^2}\right)\right|^{\epsilon+2}}{\left(1-\mu_{(1-\epsilon)e^{it^2}}\right)^2}(1-\epsilon)dt^2$$

$$\leq \frac{C_{1+\epsilon}}{\epsilon^{1-\varepsilon_{(1+\epsilon)}}}\int_{\gamma_{(1-\epsilon)}\cap\Delta_{\gamma}^{22}}\left(\int_{\mu_{(1-\epsilon)e^{it^2}(1-\epsilon)}}^{(1-\epsilon)}\sum_{j}\left|(f_j^2)'\left((\tfrac{1}{2}+\epsilon)e^{it^2}\right)\right|^2 d(\tfrac{1}{2}+\epsilon)\right)(1-\epsilon)dt^2 \leq \frac{C_{1+\epsilon}}{\epsilon^{1-\varepsilon_{(1+\epsilon)}}}\int_{(\frac{1}{2}+\epsilon)_{(1-\epsilon)}}\sum_{j}\left|(f_j^2)'\left((\tfrac{1}{2}+\epsilon)e^{it^2}\right)\right|^2(\tfrac{1}{2}+\epsilon)\,d(\tfrac{1}{2}+\epsilon)\,dt^2$$

$$\leq \frac{C_{1+\epsilon}}{\epsilon^{1-\varepsilon_{(1+\epsilon)}}}\int_{(\frac{1}{2}+\epsilon)_{(1-\epsilon)}}\sum_{j}\left|(f_j^2)'(z-\epsilon)\right|^2 dA(z-\epsilon),$$

Where

$$S_{(1-\epsilon)} \coloneqq \left\{(z-\epsilon)\in\mathbb{D}: 0\leq|z-\epsilon|\leq(1-\epsilon)\text{ and }\frac{z-\epsilon}{|z-\epsilon|}\in\gamma\right\}.$$

The proof is therefore completed.

The last result that we need before giving the proof of Theorem (2.1) is the following one (see Brahim Bouya, 2008).

Lemma (4.8):

$$\int_{\Delta_\gamma^{22}}\sum_{j}\left|f_j^2(z)\right|^{2(1+\epsilon)} a_\gamma^2(z)dA(z)\leq\sum_{j}C_{1+\epsilon}\left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)}^2+CA(\Delta_\gamma).$$

Proof: Using (19) and Lemmas (4.6) and (4.7), we find that

$$\int_{\Delta_\gamma^{22}} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} a_\gamma^2(z)dA(z)$$

$$= \frac{1}{\pi}\int_0^1 \left(\int_{\gamma_{(1-\epsilon)} \cap \Delta_\gamma^{22}} \sum_j \left|f_j^2\left((1-\epsilon)e^{it^2}\right)\right|^{2(1+\epsilon)} a_\gamma^2\left((1-\epsilon)e^{it^2}\right)(1-\epsilon)dt^2 \right) d(1-\epsilon)$$

$$\le CA(\Delta_\gamma) + 2^{(2\epsilon+1)} \int_0^1 \left(\int_{\gamma_{(1-\epsilon)} \cap \Delta_\gamma^{22}} \sum_j \left| f_j^2\left((1-\epsilon)e^{it^2}\right) - f_j^2\left(\mu_{(1-\epsilon)e^{it^2}}(1-\epsilon)e^{it^2}\right)\right|^{2(1+\epsilon)} a_\gamma^2\left((1-\epsilon)e^{it^2}\right)(1-\epsilon)dt^2 \right) d(1-\epsilon)$$

$$\le CA(\Delta_\gamma) + \sum_j C_{1+\epsilon} \left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)}^2.$$

This completes the proof of the lemma.

Conclusion**.** Now, according to (18) and Lemmas (4.4), (4.5) and (4.8), we obtain

$$\int_{\gamma_{(1-\epsilon)} \cap \Delta_\gamma^{22}} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} \left|((f_j^2)_\Gamma)'(z)\right|^2 dA(z)$$

$$\le 2\sum_j \left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)}^2 + 8\int_{\gamma_{(1-\epsilon)} \cap \Delta_\gamma^{22}} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} a_\gamma^2(z)dA(z)$$

$$\le \sum_j C_{1+\epsilon}\left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)}^2 + CA(\Delta_\gamma).$$

Combining this with Lemma (4.3), we deduce that

$$\int_{\Delta_\gamma} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} \left|((f_j^2)_\Gamma)'(z)\right|^2 dA(z) \le \sum_j C_{1+\epsilon}\left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)}^2 + CA(\Delta_\gamma).$$

Hence

$$\int_{\mathbb{D}} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} \left|((f_j^2)_\Gamma)'(z)\right|^2 dA(z) = \sum_{n=1}^{\infty} \int_{\Delta_{\gamma n}} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} \left|((f_j^2)_\Gamma)'(z)\right|^2 dA(z)$$

$$\le \sum_j C_{1+\epsilon} \sum_{n=1}^{\infty} \left\|(f_j^2)'\right\|_{L^2(\Delta_{\gamma n})}^2 + C\sum_{n=1}^{\infty} A(\Delta_{\gamma n}) \le C_{1+\epsilon}.$$

This completes the proof of Theorem (2.1)

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