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Original Paper

Riemann's Hypothesis New Proof

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Abstract

After some papers proving the RC (short for "Riemann's Conjecture", also known as the "Riemann's Hypothesis", RH), now the author provides a new proof, using the "Spira Criterion" that states "The RH is equivalent to the statement that if σ>0.5 and t> 6.5 then $|\zeta(1-s)|>|\zeta(s)|$". We use the concept of "transfer function" for control systems. This new proof is so simple that the author wonders why a great mathematician like Riemann did not see it; therefore F. Galetto thinks that somewhere in the purported proof there should be an error.

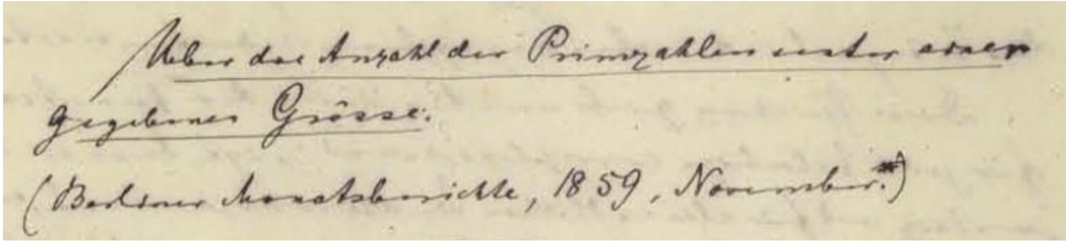
Keywords

Zeta Function, Functional Equation, Nontrivial Zeros, Critical Line, Spira Criterion, Generalised Power Transfer Function.

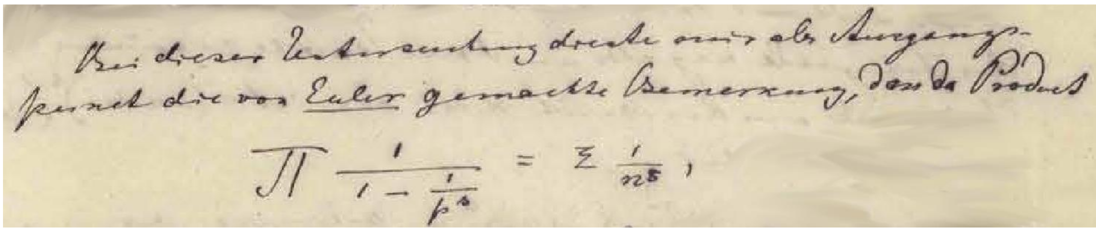
1. Introduction

It is well known that for over a century mathematicians have been trying to prove the so-called Riemann Hypothesis, RH for short, a conjecture claimed by Riemann [who was professor at University of Gottingen in Germany], near 1859 in a 8-page paper "On the number of primes less than a given magnitude" shown at Berlin Academy, and dated/published in 1859; it is well known, as well, that RH is related to set of all the Prime Numbers (Figure 1); prime numbers are fundamental for encryption of documents and data (e.g. payments with credit cards).

He invented the *Riemann zeta function*: $\zeta(z)$, where z is a complex number z=x+iy and i is the "positive" imaginary unit such that $i^2=-1$. In the following part of the paper we will use the symbols $s=\sigma+it$ for the complex variable, as in Titchmarsh (1986).


Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.
(Berliner Monatsberichte, 1859, November.*)

……………


Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s},$$

Figure 1. From the Original B. Riemann Manuscript

The description the zeta function $\zeta(s) = \zeta(\sigma + it)$ will not be introduced here, interested readers can refer to any text book, e.g., Titchmarsh (1986), Bombieri (2000), Broughan (2017).

$\zeta(s)$ is a meromorphic function with only a simple pole at s=1 [with residue 1]; moreover $\zeta(s) \neq 0$ for all z ∊ ***C*** with Re(s)=σ>1; it has zeros at the negative even integers **-2, -4, -6, …..**, named ***trivial zeros***. The other zeros are named ***nontrivial*** *zeros*: all the *"known"* zeros, computed up to now [up to 2004, 10^{12} zeros have been computed, all on the *Critical Line*], are the complex numbers s=1/2 + it, with suitable values of t.

It satisfies the functional equation

$$\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\,\Gamma[(1-s)/2])\zeta(1-s) \tag{1}$$

The properties of ζ(s) are as follows:

a) ζ(s) has no zero for Re(s)>1;

b) the only pole ζ(s) is at s=1: it is simple and has residue 1;

c) ζ(s) has trivial zeros at the negative even integers z = -2, -4, -6, ….

d) all the nontrivial zeros lie inside the region, named *Critical Strip*, 0 ≤ Re(s) ≤ 1 and are symmetric about both the vertical line, named *Critical Line*, Re(z)=σ=1/2 and the real axis Im(z)=0: $\zeta(s) = \overline{\zeta(s)} = \zeta(\bar{s}) = \zeta(1-s) = \zeta(1-\bar{s})$

Riemann conjectured the so-called Riemann Hypothesis (RH), or Conjecture (RC): *RC* states *All nontrivial zeros of ζ(s) have real part σ equal to 1/2 .*

Many great mathematicians tackled this problem; we do not mention them, because they can be found in many books and papers.

If RC would be related to Physics, it would be considered a "universal law": up to 2004, 10^{12} zeros have been computed, all on the *Critical Line*.

If RC would be related to Statistics, <<the hypothesis H_0: "the nontrivial zeros are on the *Critical Line*">>, would be confirmed with a Confidence Level (CL) > 0.9999999999: the evidence of 10^{12} zeros computed, all on the *Critical Line* (as to 2004) supports H_0 with that "high" CL.

But Mathematics asks much more than Statistics and Physics…

Also a theorem of G. Hardy [*Hardy's Theorem, 1914*] that proved that "*There are infinitely many zeros of ζ(s) on the Critical Line*" is not enough.

ALL the nontrivial zeros must be on the *Critical Line*, if one wants to prove RC.

The author, Fausto Galetto, is aware that (in these weeks) he has been affording a very important problem that great mathematicians have failed to prove.

2. The "Transfer Function" and the Spira Criterion

From (1) we derive

$$\zeta(1-s) = \pi^{0.5-s}\,\frac{\Gamma(s/2)}{\Gamma[(1-s)/2])}\zeta(s) \tag{2a}$$

which can be written, in the form used in electronics, communication theory and control systems,

$$\zeta(1-s) = H(s)\zeta(s) \tag{2b}$$

where H(s) is the "*Transfer Function*" that links the *Input* $\zeta(s)$ to the *Output* $\zeta(1-s)$.

The "*Transfer Function*" is the ratio *Output/Input*.

When σ=0 the "*Transfer Function*" depends only on the "frequency" ω; from now on we use the symbol s=σ+jω, as it is customary in electronics, communication theory and control systems:

$$\zeta(1-\sigma-j\omega) = H(\sigma+j\omega)\zeta(\sigma+j\omega) \tag{2c}$$

When σ=0 we have

$$\zeta(1-j\omega) = H(j\omega)\zeta(j\omega) \tag{2d}$$

Squaring both side and taking the absolute value, we get

$$|\zeta(1-j\omega)|^2 = |H(j\omega)|^2|\zeta(j\omega)|^2 \tag{3}$$

This formula is the same as the following

$$S_{Output}(j\omega) = |H(j\omega)|^2 S_{Input}(j\omega) \tag{3b}$$

used in electronics, communication theory and control systems, where the function S(jω) is the "Power Spectral Density". We name here $|H(j\omega)|^2$ the "Power Transfer Function".

In analogy to (3b) we define

$$|\zeta(1-s)|^2 = |H(s)|^2|\zeta(s)|^2 \tag{3c}$$

and name here $|H(s)|^2$ the "Generalised Power Transfer Function".

The *Spira Criterion* (Broughan, 2017) for RC is

The Riemann Hypothesis is equivalent to the statement that

$|\zeta(1-s)| > |\zeta(s)|$ *for any 0.5<σ<1 and any* $\omega \geq 2\pi + 0.1$.

Notice that $|H(s)| > 0$ for any s and $|H(0.5+j\omega)| = 1$ because $|H(1-s)H(s)| = 1$

Since $|\zeta(1-s)/\zeta(s)| = |\mathrm{H(s)}|$, the *Spira Criterion* (Broughan, 2017) for RC states that $|\mathrm{H}(\sigma+j\omega)| > 1$ *for any 0.5<σ<1 and any* $\omega \geq 2\pi + 0.1$.

We can use the infinite product expansion for the Gamma Function and we get

$$|\mathrm{H(s)}| = |\pi^{0.5-s}| \left[\frac{\left|\Gamma\left(\frac{s}{2}\right)\right|}{\left|\Gamma\left(\frac{1-s}{2}\right)\right|}\right] = |\pi^{0.5-s}| |e^{\gamma(0.5-s)}| \left|\frac{1-s}{s}\right| \prod_1^\infty \left\{\frac{\left|\left(1+\frac{1-s}{2n}\right)\right|}{\left|\left(1+\frac{s}{2n}\right)\right|} \left|e^{\frac{s-0.5}{n}}\right|\right\} =$$

$$|\pi^{0.5-s}| |e^{\gamma(0.5-s)}| \left|\frac{1-s}{s}\right| \prod_1^\infty \left\{\frac{|(2n+1-s)|}{|(2n+s)|} \left|e^{\frac{s-0.5}{n}}\right|\right\} \quad (4)$$

Figure 2 depicts one of the "products" within the braces

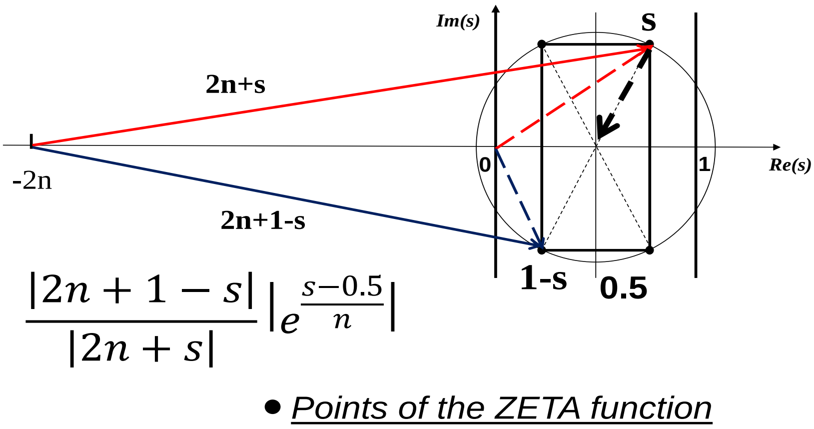


Figure 2. Picture of One of the "Products" within the Braces

We can also find the "Generalised Power Transfer Function"

$$|\mathrm{H(s)}|^2 = |\pi^{0.5-s}|^2 \left|\left[e^{-\gamma(s-0.5)}\right]\right|^2 \left|\frac{1-s}{s}\right|^2 \prod_1^\infty \left\{\frac{|(2n+1-s)|^2}{|(2n+s)|^2} \left|e^{\frac{s-0.5}{n}}\right|^2\right\} \quad (5)$$

This can be written as a real function A(σ,ω)

$$\mathrm{A}(\sigma,\omega) = \pi^{1-2\sigma} e^{-\gamma(2\sigma-1)} \frac{(1-\sigma)^2+\omega^2}{\sigma^2+\omega^2} \prod_1^\infty \left\{\frac{(2n+1-\sigma)^2+\omega^2}{(2n+\sigma)^2+\omega^2} e^{\frac{(2\sigma-1)}{n}}\right\} \quad (5b)$$

3. The Proof of RC

To prove RC, we use the *Spira Criterion* (Broughan, 2017) for RC: we show that $|\mathrm{H(s)}| = |\mathrm{H}(\sigma+j\omega)| > 1$ for any 0.5<σ<1 and any $\omega \geq 2\pi + 0.1$.

For any chosen $\omega \geq 2\pi + 0.1$, we can find a value n_0 such that, for any $n>n_0$, the logarithm is expended as $\ln[1+z/(2n)] \approx \{z/(2n) - z^2/[2(2n)^2]\}$, with $z = \sigma + j\omega$; then

$$ln\left(1+\frac{1-s}{2n}\right) - ln\left(1+\frac{s}{2n}\right) \approx \frac{(1-2s)}{2n}\left[1-\frac{1}{2(2n)}\right]$$

Therefore, for any $n>n_0$, each factor in the "infinite product" is

$$a_n = \left\{\frac{|1+(1-s)/(2n)|}{|1+s/(2n)|}\left|e^{\frac{s-0.5}{n}}\right|\right\} \approx \left|e^{(s-0.5)\left[\frac{1}{(2n)^2}\right]}\right| = e^{Re\left\{(s-0.5)\left[\frac{1}{(2n)^2}\right]\right\}} = e^{(\sigma-0.5)\left[\frac{1}{(2n)^2}\right]} > 1$$

See Figure 3

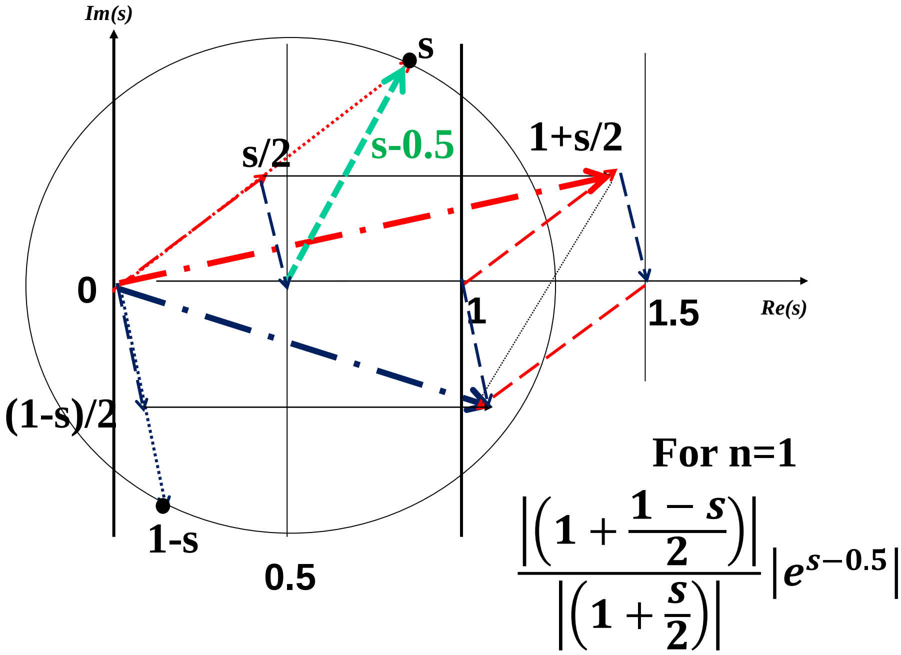


Figure 3. Picture of the First Factor within the Braces

Since $a_n > a_{n+1}$, we get $\prod_1^\infty a_n > 1$ and hence $|\mathrm{H}(s)| > 1$.

Therefore RC is proved by the Spira Criterion (Broughan, 2017).

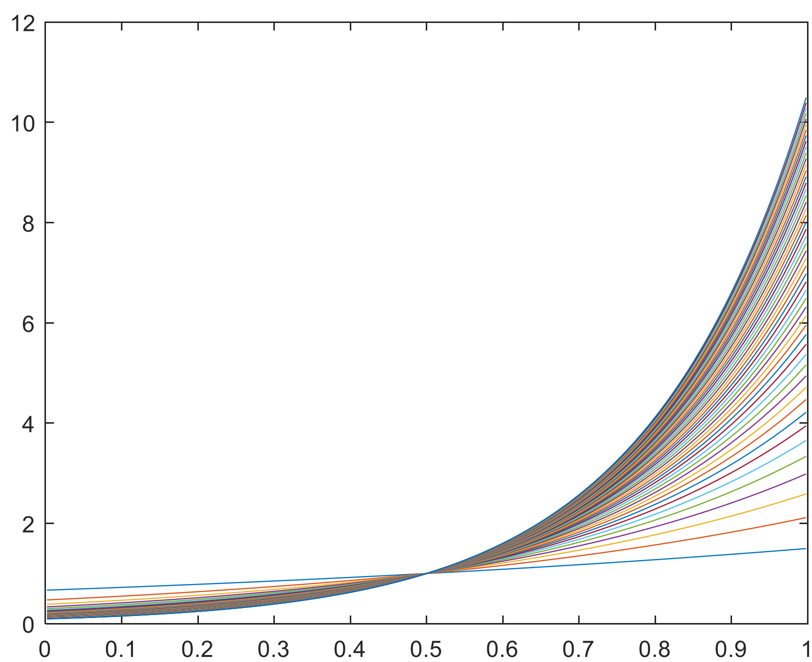


Figure 4. Behaviour of $|\mathrm{H}(\sigma + j\omega)|$, Abscissa is σ, Curves are Indexed by ω

Figure 4 show the behaviour of $|\mathrm{H}(\sigma+\mathrm{j}\omega)|$ for various values of (abscissa) σ and various ω (curves); all curves intersects at σ=0.5 and have ordinate $|\mathrm{H}(\sigma+\mathrm{j}\omega)|=1$

We can confirm the previous result by writing it in a different way (Abramowitz et al., 1965) the gamma function; so the absolute "transfer function" becomes

$$|\mathrm{H(s)}| = |\pi^{0.5-s}| \frac{\left|\Gamma\left(\frac{\sigma}{2}\right)\right|}{\left|\Gamma\left(\frac{1-\sigma}{2}\right)\right|} \sqrt{\prod_0^\infty \left[1+\left(\frac{\omega/2}{n+(1-\sigma)/2}\right)^2\right] / \left[1+\left(\frac{\omega/2}{n+\sigma/2}\right)^2\right]} \tag{4b}$$

Let's now consider the "Generalised Power Transfer Function" $|\mathrm{H(s)}|^2$; from above, we have the ***power difference*** $|\zeta(1-\mathrm{s})|^2-|\zeta(\mathrm{s})|^2=|\zeta(\mathrm{s})|^2\{|\mathrm{H(s)}|^2-1\}>0$.

Let's consider the equation

$$|\zeta(\mathrm{s})|^2\{|\mathrm{H(s)}|^2-1\}=0 \tag{6}$$

Form (6) we find two solutions

$$|\zeta(\mathrm{s})|^2=0 \tag{7}$$

$$\{|\mathrm{H(s)}|^2-1\}=0 \tag{8}$$

(7) entail that $\mathrm{s}=\sigma+\mathrm{j}\omega$ is a zero of the Zeta Function $\zeta(\sigma+\mathrm{j}\omega)$, while (8) entails that the real function A(σ,ω) equals 1 for any ω

$$\pi^{1-2\sigma} e^{-\gamma(2\sigma-1)} \frac{(1-\sigma)^2+\omega^2}{\sigma^2+\omega^2} \prod_1^\infty \left\{\frac{(2n+1-\sigma)^2+\omega^2}{(2n+\sigma)^2+\omega^2} e^{\frac{(2\sigma-1)}{n}}\right\}=1 \tag{9}$$

It is easily seen that (9) has a unique solution $\sigma=0.5$; therefore the zeros of the Zeta Function $\zeta(\sigma+\mathrm{j}\omega)$ are all on the ***Critical Line*** $s=0.5+j\omega$; this is the same result found before by the ***Spira Criterion***.

We can confirm the previous result by writing in a different way (Abramowitz et al., 1965) the gamma function; so the "Generalised Power Transfer Function" $|\mathrm{H(s)}|^2$ becomes

$$|\mathrm{H(s)}|^2 = \pi^{1-2\sigma}\left[\frac{\left|\Gamma\left(\frac{\sigma}{2}\right)\right|}{\left|\Gamma\left(\frac{1-\sigma}{2}\right)\right|}\right]^2 \prod_0^\infty \left[1+\left(\frac{\omega/2}{n+(1-\sigma)/2}\right)^2\right] / \left[1+\left(\frac{\omega/2}{n+\sigma/2}\right)^2\right]=1 \tag{9b}$$

It is easily seen that (9b) has a unique solution $\sigma=0.5$; therefore …. (as before).

4. Conclusion

Titchmarsh proved that there are infinite zeros of the *Riemann zeta function* ζ(s) in the Critical Strip, that is there are infinite values $s_k=\alpha_k+i\beta_k$ such that $\zeta(s_k)=0$, $[0<\alpha_k<1]$. G. Hardy [*Hardy's Theorem, 1914*] proved that "*There are infinitely many zeros of ζ(s) on the Critical Line*": that fact was not conclusive, because ALL the zeros must be on the Critical Line.

In this paper, to prove RC we used first the "Spira Criterion" [with the Transfer Function H(s)] and second the "Generalised Power Transfer Function" $|\mathrm{H(s)}|^2$.

The result is that ***RH (RC) is true***. (as done in previous papers, Fausto Galetto, 2014, 2018, 2019).

References

Abramowitz, M., & Stegun, I. A. (1965). Handbook of Mathematical Functions. National Bureau of Standards, Applied Math. Series #55, Dover Publications, sec. 6.5.

Bombieri, E. (2000). *Problems of the Millennium: The Riemann Hypothesis*. Retrieved from http://claymath.org/prizeproblems/riemann.htm

Broughan, K. (2017). *Equivalents of the Riemann Hypothesis, Arithmetic Equivalents, Encyclopedia of Mathematics and Its Applications*. Cambridge University Press.

Fausto Galetto. (2014). Riemann Hypothesis proved. *Academia Arena*, *6*(12), 19-22.

Fausto Galetto. (2018). A new proof of the Riemann Hypothesis. *Research Trends on Mathematics and Statistics*, *3*, 23-35, 2019 and HAL archive, 2018.

Fausto Galetto. (2019). Riemann's Conjecture, a "One Page Proof (new)". *HAL archive, 2019* and *Academia.edu 2019*.

Luigi Amerio. (1989). Analisi Matematica, *III*, parte 1°, UTET.

Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta-Function*. Clarendon Press, Oxford.

Wolfram MathWorld, Xi function. (2019).