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Original Paper

Vandermonde Determinant and Its Applications

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Abstract

Vandermonde determinant is an important object in linear algebra, which has special and elegant structure, and admits the significant applications in many different areas. In this paper, we summarize two typical proofs and present many applications of Vandermonde determinant.

Keywords

Vandermonde determinant, proofs, applications

1. Introduction

Vandermonde determinant was named after Alexandre-Theophile Vandermonde, who is believed to be the founder of determinant theory. Vandermonde determinant has special and elegant structure (Han, 2020; Vijayalakshmi, Bulut, & Sudharsan, 2022). For a positive integer $n \geq 2,$ the Vandermonde determinant of order *n* is defined as follows:

$$V_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_n^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}.$$

When $n = 2,$ V_2 is obviously equal to $x_2 - x_1,$ and when $n = 3,$ we have

$$V_3 = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1).$$

Based on this two terms, it is easy to conjecture that $V_n = \prod_{1\le j<i\le n}(x_i - x_j)$, where $n \ge 2$, and

$$\prod_{1\le j<i\le n}(x_i - x_j) = (x_2 - x_1)(x_3 - x_1)\cdots(x_n - x_1)(x_3 - x_2)\cdots(x_n - x_2)\cdots(x_n - x_{n-1}).$$

In fact, this conjecture has been proved in many different ways, and we will select some typical proofs in the next section.

Vandermonde determinant are significant in many different areas, due to their important applications. For instance, Vandermonde determinant can be used in linear algebra for determining the number of roots of polynomials with degree n, for determining the value of some special determinants (Li & Ma, 2017), for proving whether a linear transform is reversible (Han, 2020; Pan, 2022), for determining the linear dependence of vectors (Zhang & Chen, 2020); Vandermonde determinant can also be used in calculus (Pan, 2022; Zhang & Chen, 2020), and in digital signal processing and approximation problems (Vijayalakshmi, Bulut, & Sudharsan, 2022).

In this paper, we focus our attention on the proofs and applications of Vandermonde determinant. Firstly, we summarize two typical proofs of Vandermonde determinant. Then, we show several applications of Vandermonde determinant in linear algebra. The results of this paper will strengthen the understanding of Vandermonde determinant, and will be helpful for study of linear algebra.

2. Two Typical Proofs of Vandermonde Determinant

The value of Vandermonde determinant has been introduced in Introduction. Now we restated it in the following theorem.

Theorem 2.1 Let V_n be the Vandermonde determinant of order n. Then we have

$$V_n = \prod_{1\le j<i\le n}(x_i - x_j),$$

where $n \ge 2$ is a positive integer.

By Theorem 2.1, it is clearly that the sufficient and necessary condition of $V_n \ne 0$ is that

$x_1, x_2, \cdots, x_n$ are mutually different.

Now we shall present two typical proofs of Theorem 2.1. The first proof is obtained from the perspective of row elementary transformation (Wang & Shi 2013), which is given as follows.

Proof .(**Method 1** from the row elementary transformation)

We prove Theorem 2.1 by induction on the order n of Vandermonde determinant. Firstly, the result is clearly true for $n = 2$.

Then, assume that the result holds for $n-1$. Now we prove that Theorem 2.1 holds for n. Based on the properties of determinants (see Wang & Shi, 2013), from line $n-1,$ multiply by x_1 for the elements of every line, and add them to the elements of next line, we obtain that

$$V_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & x_2 - x_1 & x_3 - x_1 & \cdots & x_n - x_1 \\ 0 & x_2(x_2 - x_1) & x_3(x_3 - x_1) & \cdots & x_n(x_n - x_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & x_2^{n-2}(x_2 - x_1) & x_3^{n-2}(x_3 - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix}$$

$$= \begin{vmatrix} x_2 - x_1 & x_3 - x_1 & \cdots & x_n - x_1 \\ x_2(x_2 - x_1) & x_3(x_3 - x_1) & \cdots & x_n(x_n - x_1) \\ \vdots & \vdots & & \vdots \\ x_2^{n-2}(x_2 - x_1) & x_3^{n-2}(x_3 - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix}$$

$$= (x_2 - x_1)(x_3 - x_1)\cdots(x_n - x_1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \end{vmatrix}.$$

The last determinant in the above is a Vandermonde Determinant of order $n-1,$ by induction, we have

$$V_n = (x_2 - x_1)(x_3 - x_1)\cdots(x_n - x_1) \prod_{2 \le j < i \le n}(x_i - x_j) = \prod_{1 \le j < i \le n}(x_i - x_j).$$

This completes the proof.

Now we provide the second proof of Vandermonde determinant, which is obtained from the perspective of column elementary transformation (Zhang & Yin, 2013).

Proof. (**Method 2** from the column elementary transformation)

We also prove Theorem 2.1 by induction on the order n of Vandermonde determinant. Firstly, the result is true clearly for $n=2.$ Then assume that the result holds for $n-1.$ Below, we prove the case of $n.$ Adding the negative elements of the first column to the rest of columns of V_n , we obtain that

$$V_n = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ x_1 & x_2 - x_1 & x_3 - x_1 & \cdots & x_n - x_1 \\ x_1^2 & x_2^2 - x_1^2 & x_3^2 - x_1^2 & \cdots & x_n^2 - x_1^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} - x_1^{n-1} & x_3^{n-1} - x_1^{n-1} & \cdots & x_n^{n-1} - x_1^{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} x_2 - x_1 & x_3 - x_1 & \cdots & x_n - x_1 \\ x_2^2 - x_1^2 & x_3^2 - x_1^2 & \cdots & x_n^2 - x_1^2 \\ \vdots & \vdots & & \vdots \\ x_2^{n-1} - x_1^{n-1} & x_3^{n-1} - x_1^{n-1} & \cdots & x_n^{n-1} - x_1^{n-1} \end{vmatrix}$$

$$= (x_2 - x_1)(x_3 - x_1)\cdots(x_n - x_1)D_{n-1,}$$

where

$$D_{n-1}=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2+x_1 & x_3+x_1 & \cdots & x_n+x_1 \\ x_2^2+x_2x_1+x_1^2 & x_3^2+x_3x_1+x_1^2 & \cdots & x_n^2+x_nx_1+x_1^2 \\ \vdots & \vdots & & \vdots \\ \sum\limits_{i+j=n-2} x_2^i x_1^j & \sum\limits_{i+j=n-2} x_3^i x_1^j & \cdots & \sum\limits_{i+j=n-2} x_n^i x_1^j \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ x_1 & 1 & 0 & \cdots & 0 \\ x_1^2 & x_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-2} & x_1^{n-3} & x_1^{n-4} & \cdots & 1 \end{vmatrix}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \end{vmatrix}.$$

By induction, we have $D_{n-1}=\prod\limits_{2\le j<i\le n}(x_i-x_j)$. Therefore, $V_n=\prod\limits_{1\le j<i\le n}(x_i-x_j)$. The proof is completed.

Comments on the above two proofs:

Compared the above two proofs, the second proof is clearly more complex than the first one, which depends on a matrix factorization that is not easily obtained in general. The first proof is common, which was adopted by many linear algebra books. In addition, we find that both two proofs are given based on mathematical induction, which is a classical method in the proof of mathematics. Many mathematical theorem can be proved by this method. The key of this method is to find the recurrence relations of V_n and V_{n-1}. Hence, simplifying V_n to V_{n-1} is important in the proof.

3. Several Applications of Vandermonde Determinant

In this section, we present several applications of Vandermonde Determinant. The first application is the use of Vandermonde Determinant in determining the number of roots of polynomials with degree n, which is given as follows.

Theorem 3.1 Let $f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$ be a polynomial over complex field C, where $a_0,a_1,\cdots,a_n\in C$ are not zero at the same time. Then the number of roots of $f(x)=0$ is at most n.

Proof. Assume that there are $n+1$ mutually distinct $x_1, x_2, \cdots, x_{n+1} \in C$ such that $f(x_i)=0$ for each $i \in \{1,2,\cdots,n+1\}$, that is, the following system with respect to $a_0, a_1, a_2 \cdots, a_n$ holds:

$$\begin{cases} a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_n x_1^n = 0 \\ a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_n x_2^n = 0 \\ \vdots \\ a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + \cdots + a_n x_{n+1}^n = 0 \end{cases}.$$

But the coefficient matrix of the above system is the transposition of a Vandermonde matrix, whose determinant is not zero by Theorem 2.1. By Cramer's rule, we obtain that $a_i = 0$ for each $i \in \{1,2,\cdots,n+1\}$, which contradicts the assumption. This completes the proof.

Comments to Theorem 3.1: Theorem 3.1 is a classical result in "complex variable function", which is very useful for characterizing the number of roots of a polynomial. Here, a simple proof is given by using Vandermonde determinant.

Another application of Vandermonde determinant in the theory of polynomial is to determine the values of some special polynomials by Vandermonde determinant, see the following result.

Theorem 3.2 Let $p_1, p_2, \cdots, p_{n-1}$ be $n-1$ polynomials over the complex field C, such that $p_1(x^n) + xp_2(x^n) + \cdots + x^{n-2} p_{n-1}(x^n)$ is divisible by $1 + x + x^2 + \cdots + x^{n-1}$. Then $p_1(1) = p_2(1) = \cdots = p_{n-1}(1) = 0.$

Proof. By assumption, assume that

$$p_1(x^n) + xp_2(x^n) + \cdots + x^{n-2} p_{n-1}(x^n) = p(x)(1 + x + x^2 + \cdots + x^{n-1})$$

for some polynomial p. Let $\omega = e^{\frac{2\pi}{n}i}$. Then we have $\omega^n = 1$. Take as $x = \omega, \omega^2, \cdots, \omega^{n-1}$, respectively, Then we have $x^n = 1$ and

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1} = 0.$$

Therefore, we have

$$\begin{cases} p_1(1)+\omega p_2(1)+\cdots+\omega^{n-2}p_{n-1}(1)=0 \\ p_1(1)+\omega^2 p_2(1)+\cdots+\omega^{2(n-2)}p_{n-1}(1)=0 \\ \vdots \\ p_1(1)+\omega^{n-1}p_2(1)+\cdots+\omega^{(n-1)(n-2)}p_{n-1}(1)=0 \end{cases},$$

which is a linear system with $p_1(1), p_2(1),\cdots, p_{n-1}(1),$ whose coefficient matrix is

$$\begin{pmatrix} 1 & \omega & \cdots & \omega^{n-2} \\ 1 & \omega^2 & \cdots & \omega^{2(n-2)} \\ \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \cdots & \omega^{(n-1)(n-2)} \end{pmatrix}.$$

The determinant of the above matrix is a transposition of a Vandermonde determinant, and hence it is not equal to zero. By Cramer's rule, we have

$$p_1(1)=p_2(1)=\cdots=p_{n-1}(1)=0.$$

This completes the proof.

Comments to Theorem 3.2: In this theorem, we determine some special value of some particular polynomials via Cramer's rule and Vandermonde determinant.

Now we provide the third application of Vandermonde determinant, which is used to prove whether a linear transform is reversible by means of Vandermonde determinant.

Theorem 3.3 Let $\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ be a basis of vector space V. Let σ be a linear transformation of V such that $\sigma\alpha_i=\alpha_1+k_i\alpha_2+k_i^2\alpha_3+\cdots+k_i^{n-1}\alpha_n$ for each $i\in\{1,2,\cdots,n\}$. Then σ is reversible.

Proof. Since $\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ be a basis of V with $\sigma\alpha_i=\alpha_1+k_i\alpha_2+\cdots+k_i^{n-1}\alpha_n,$ we have $k_i\neq k_j$ for any distinct $i, j\in\{1,2,\cdots,n\},$ and $\sigma(\alpha_1,\alpha_2,\cdots,\alpha_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)D,$ where

$$D=\begin{pmatrix} 1 & 1 & \cdots & 1 \\ k_1 & k_2 & \cdots & k_n \\ \vdots & \vdots & & \vdots \\ k_1^{n-1} & k_2^{n-1} & \cdots & k_n^{n-1} \end{pmatrix}.$$

By Theorem 2.1, we have $|D| = \prod_{1 \le j < i \le n}(k_i - k_j) \neq 0.$ Hence, σ is a reversible linear transformation. The proof is finished.

Comments to Theorem 3.3: In this theorem, we judge whether a linear transformation of a vector space is reversible by Vandermonde determinant.

In the following theorem, we give the fourth application of Vandermonde determinant, which is used to determine the linearly independent of vectors by Vandermonde determinant.

Theorem 3.4 Let P be a number field, and let V be a vector space of dimension n over $P.$ Then for any positive inter $m \ge n,$ there are m vectors of V such that any n of them are linear independent.

Proof. Since V is isomorphic to $P^n,$ we consider the proof in P^n only. Now we take m vectors of P^n as follows:

$$\begin{aligned} &v_1 = (1, a, a^2, \cdots, a^{n-1}), \\ &v_2 = (1, a^2, (a^2)^2, \cdots, (a^{n-1})^2), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &v_m = (1, a^m, (a^2)^m, \cdots, (a^{n-1})^m), \end{aligned}$$

where $0 \neq a \in P.$ Then for any $1 \le k_1 < k_2 < \cdots < k_n \le m,$ the determinant

$$|(v_{k_1}, v_{k_2}, \cdots, v_{k_n})| = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a^{k_1} & a^{k_2} & \cdots & a^{k_n} \\ \vdots & \vdots & & \vdots \\ (a^{n-1})^{k_1} & (a^{n-1})^{k_2} & \cdots & (a^{n-1})^{k_n} \end{vmatrix},$$

which is a Vandermonde determinant whose value is not zero by Theorem 2.1. Hence, $v_{k_1}, v_{k_2}, \cdots v_{k_n}$ are linearly independent. The proof is finished.

Comments to Theorem 3.4: It is in general hard to directly deal with the problems of vector space. In this theorem, we simplify the problem by employing Vandermonde determinant.

The last application of Vandermonde determinant in this paper is given as follows, which is also use Vandermonde determinant to determine the linearly independence of vectors.

Theorem 3.5 Let A be a matrix of order $n.$ Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be m eigenvalues of $A.$ Let $\xi_{i1}, \xi_{i2}, \cdots, \xi_{it_i}$ are linearly independent eigenvectors of A corresponding to $\lambda_i (i = 1, 2, \cdots, m).$ Then the following vectors

$$\xi_{11}, \xi_{12}, \cdots, \xi_{1t_1}, \xi_{21}, \xi_{22}, \cdots, \xi_{2t_2}, \cdots, \xi_{m1}, \xi_{m2}, \cdots, \xi_{mt_m}$$

are linearly independent.

Proof. Assume that there are complex numbers $k_{11}, k_{12}, \cdots, k_{1t_1}, \cdots, k_{m1}, k_{m2}, \cdots, k_{mt_m}$ such that

$$\textbf{(1)}\quad k_{11}\xi_{11} + k_{12}\xi_{12} + \cdots + k_{1t_1}\xi_{1t_1} + \cdots + k_{m1}\xi_{m1} + k_{m2}\xi_{m2} + \cdots + k_{mt_m}\xi_{mt_m} = 0.$$

Let $\eta_i = k_{i1}\xi_{i1} + k_{i2}\xi_{i2} + \cdots + k_{it_i}\xi_{it_i}, i = 1,2,\cdots,m.$ Then (1) becomes that

$$(2)\quad \eta_1 + \eta_2 + \cdots + \eta_m = 0.$$

Since $\xi_{ij}(j = 1,2,\cdots,t_i)$ is a eigenvectors of A corresponding to $\lambda_i(i = 1,2,\cdots,m),$ we have $A\xi_{ij} = \lambda_i\xi_{ij}, (i = 1,2,\cdots,m;\ j = 1,2,\cdots,t_i).$ Therefore, it holds that

$$A\eta_i = k_{i1}A\xi_{i1} + k_{i2}A\xi_{i2} + \cdots + k_{it_i}A\xi_{it_i} = \lambda_i\eta_i, i = 1,2,\cdots,m.$$

Multiplying $A, A^2, \cdots, A^{m-1}$ to both sides of (2), respectively, we obtain that

$$\begin{cases} \eta_1 + \eta_2 + \cdots + \eta_m = 0 \\ \lambda_1\eta_1 + \lambda_2\eta_2 + \cdots + \lambda_m\eta_m = 0 \\ \vdots \\ \lambda_1^{m-1}\eta_1 + \lambda_2^{m-1}\eta_2 + \cdots + \lambda_m^{m-1}\eta_m = 0 \end{cases},$$

which is a linear system with $\eta_1, \eta_2, \cdots, \eta_m$, whose coefficient matrix is

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \vdots & \vdots & & \vdots \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \cdots & \lambda_m^{m-1} \end{pmatrix}.$$

The determinant of the above matrix is a Vandermonde determinant, and hence it is not equal to zero. By Cramer's rule, we have

$$\eta_1 = \eta_2 = \cdots = \eta_m = 0.$$

In addition, by the assumption that $\xi_{i1}, \xi_{i2}, \cdots, \xi_{it_i}$ are linearly independent eigenvectors of A corresponding to $\lambda_i (i = 1, 2, \cdots, m)$, we have

$$k_{11} = k_{12} = \cdots = k_{1t_1} = \cdots = k_{m1} = k_{m2} = \cdots = k_{mt_m} = 0.$$

The proof is completed.

Comments to Theorem 3.5: This theorem is a classical result in linear algebra. Many linear algebra books proved this result by induction. Here, we provide another new proof by Vandermonde determinant.

4. Conclusion

Vandermonde determinant plays a significant role in linear algebra. In this paper, we summarized two typical proofs of Vandermonde determinant, and presented its several applications, which will strengthen the understanding of Vandermonde determinant, and contribute to the study of linear algebra.

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