

Original Paper

How We Understand Hilbert's Thought of Infinity

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Abstract

We know that Hilbert's thought of infinity has profoundly influenced and changed the mathematical development of the 20th century, and yet there is inherent contradiction in his thought of infinity itself, building his understanding of infinity on Kant's intuition and the principle of finalism. This paper analyzes his thought of infinity based on Hegel's view of dialectical infinity, and points out the incompleteness of his understanding of infinity.

Keywords

Hilbert, the view of dialectical infinity

Hilbert, we know, is one of the greatest mathematicians in modern human history. At the Second International Congress of Mathematicians in Paris (1900), he proposed 23 mathematical problems that mathematicians should work hard in the new century, which were considered the highest point of mathematics in the 20th century. Research on these issues strongly promoted the development of mathematics in the 20th century and had a profound influence in the world. The school of mathematics under Hilbert was a banner of the mathematical world of the late 19th and early 20th century, hence Hilbert was called *the uncrowned king of mathematics*, who was the genius of genius.

Hilbert's influence on the mathematical world is also particularly reflected in his understanding of infinity, and his thought has profoundly influenced and changed the course of the mathematical development of the world. In his article *On the infinite* there was an exciting story: "From time immemorial, the infinite has stirred men's *emotions* more than any other question. Hardly any other *idea* has stimulated the mind so fruitfully. Yet, no other *concept* needs *clarification* more than it does." (Benacerraf & Putnam, 2003, p. 212) However, there is self-contradiction in his thought of infinity. How can we correctly understand his thought of infinity?

The contradiction of Hilbert's thought of infinity is mainly manifested in: On the one hand, he acknowledged the existence of *actual infinity* (objective or logical) and legal status, fully affirming

Cantor's theory of *actual infinity*. He noted that: "No one can expel us from the heaven Cantor created to us" (Benacerraf & Putnam, 2003, p. 219); "But it may still be the case that the infinite occupies a justified place *in our thinking*, that it plays the role of an indispensable concept." (Benacerraf & Putnam, 2003, p. 214); "This theory is, I think, the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity." (Benacerraf & Putnam, 2003, p. 216). On the other hand, he believes that we cannot find infinity in the real world, and that we can only master infinity through finiteness. He argued that: "The attempt to prove the infinity of space by pure speculation contains gross errors." (Benacerraf & Putnam, 2003, pp. 213-214); **"We have seen before that, whatever experience, observation and knowledge, nowhere to find infinity in the real world."** (Benacerraf & Putnam, 2003, p. 220)

At the same time, Hilbert held high the banner of mathematical certainty, proposing that **"We must know, we will know"**, believing that through finite methods we can fully master the truth. He noted in his article, "Just as operations with the infinitely small were replaced by operations with the finite which yielded exactly the same results and lead to exactly the same elegant formal relationships, so in general must deductive methods based on the infinite be replaced by finite procedures which yield exactly the same results; i.e., which make possible the same chains of proofs and the same methods of getting formulas and theorems." (Benacerraf & Putnam, 2003, p. 211) Therefore, the contradiction of Hilbert's thought of infinity deeply reflects the core contradiction between the subjects and objects, namely, the contradiction between finite and infinite.

So how do we understand his contradictory thought of infinity?

1. Why Does Hilbert Adhere to the Principle of Finalism to Solve the Problem of Infinite?

Hilbert is known to stick to solving the problem of infinite with the principle of finalism. The author analyzed it for the following reasons:

First, he sees that Weierstrass laid a rigorous and firm logical foundation for mathematical analysis, solved various difficulties arising by the concept of infinitesimal, and recognizes that *potential infinite* as a finite method is a useful and reliable tool to grasp infinity. He noted in his article that, "Weierstrass's *analysis* did indeed eliminate the infinitely large and the infinitely small by reducing statements about them to [statements about] relations between finite magnitudes." (Benacerraf & Putnam, 2003, p. 210)

Secondly, as a loyal believer of Kant's philosophy, he believes that infinity cannot be found in the physical real world, which is an illusion that exists outside experience as an ideal concept, and something that relies on intuition. He said: "So too we must realize that the infinite in the sense of an infinite totality, where we still find it used in deductive methods, is an illusion." (Benacerraf & Putnam, 2003, p. 211) Therefore, in his opinion, the *infinite whole* is an illusion, can only appear as an ideal concept, the *actual infinity* does not belong to the category of possible experience. It is this

unexperience that determines the difficulty of our human understanding infinite, leading to different schools to put forward different understanding methods and ways, including materialism and idealism. Moreover, from the artistic point of view, he did not want to give up the idea of *actual infinity*, because he saw the beauty of the foundation of mathematical analysis (strictly transformed by Weierstrass) and Cantor's theory of infinite set, and saw the **indispensability** of *actual infinity* to mathematical theory. He so appreciated the beauty of mathematical analysis that in *On the Infinite* he wrote: "In a certain sense, mathematical analysis is a symphony of the infinite." (Benacerraf & Putnam, 2003, p. 215); "But it may still be the case that the infinite occupies a justified place in our thinking, that it plays the role of an indispensable concept." (Benacerraf & Putnam, 2003, p. 214)

Finally, in the depths of his mind, the world is recognizable, and the certainty of mathematics is an unbroken philosophical belief. He said: "The goal of my theory is to establish once and for all the certitude of mathematical methods." (Benacerraf & Putnam, 2003, p. 211); "It should be seen that as mathematicians, we are standing on the top of the mountains of precise scientific research. We have no choice but to assume this noble duty in duty-bound." (Constance Reid, p. 202) Throughout Hilbert's life, he has been committed to finding the general laws in mathematics (such as his many major contributions to mathematical science: invariant theory, geometric foundation, number theory report, integral equations, mathematical foundation, etc.), and firmly believes in the organic unity of mathematical science. This idea necessarily drove him to find ways of mastering the mysteries of infinite; in his opinion, this *ideal infinity* is accessible to mankind, and some examples can prove it (such as the limit problem of infinite sequence). Because he failed to grasp Hegel's dialectic thought, it inevitably brought him back to the thought of finalism (*potential infinity*).

2. To Understand Hilbert's Contradictory Thought of Infinite Is Actually Very Simple, That Is, to Adhere to Hegel's View of Dialectical Infinity

As we know, Mr. Hegel's greatest contribution is to propose the idea of dialectic, especially the idea of *dialectical infinite* (real infinite, bad infinite) in mathematics. Based on this idea, Hegel has made a correct philosophical elaboration on the concept of limit, and revealed the thought of *bad infinity* and *real infinity* in mathematics from the law of the mutual change of quality and quantity, strongly criticized the metaphysical ideological trend in mathematics, let us fully understand the essence of *limit*, and provided a reliable philosophical basis for the thorough solution of the second mathematical crisis. Mr. Engels, the great philosopher, carried forward this thought of dialectical infinity. What is dialectical infinite, please refer to the article *On the Differences between Philosophical Infinity and Mathematical Infinity* (published in issues 2 and 3 of Journal of Educational Institute of Jilin Province, 2018), and *Philosophical Infinity and Mathematical Infinity—On the fundamental error of Actual infinity* (published in the British Philosophy of Mathematics Education Journal No.35, in charge of the noted mathematical philosopher Paul Ernest).

2.1 What Is the View of Dialectical Infinity

The brief introduction is as follows: Hegel believes that the infinite thing has a double meaning, infinite thing is the unity of *bad infinite* and *real infinite*; not only confirms that the *bad infinity* is a basic form of infinity, but also criticizes the one sidedness, which only talks about the *bad infinity* and does not pay attention to the *real infinity*. So infinity is the unity of *bad infinity* and *real infinity*, it is a *free self Being*; *Real infinity* cannot be separated from *bad infinity*, *Being-for-self* is inseparable from *Being-in-itself*. Human understanding of the infinite, from possible to reality, from the abstract to the concrete, has completed the transformation from *bad infinity* to *real infinity*. From the cognitive process, it can be said that the *bad infinity* is the result of people's understanding of the infinite things from outside, from the phenomenon, while the *real infinity* is the product of people's understanding of infinity, which is based on the fact that people go deep into the interior of things, from the general connection of things, in essence. The transition from *bad infinity* to *real infinity* marks the deepening of human cognition, reflecting the development process of human understanding of the infinite from possibility to reality, from abstract to concrete, reflects the subjective initiative of human beings in understanding the problem of infinity.

The *real infinity* is present, concrete, completed infinite, is *Being-for-self* and rational *Being*, is the completed quality; and the *bad infinity* is possible, abstract, uncompleted infinite, is *Being-in-itself* and intellectual *Being*. The difference between *real infinity* and *bad infinity* reflects the opposition between dialectical understanding and metaphysical understanding. Hegel concludes in *The Logic of Hegel* that: "But by Dialectic is meant an indwelling tendency outwards and beyond; by which the one-sidedness and limitation of the formulae of understanding is seen in its true light, and shown to be the negation of these formulae. Things are finite, just because they involve their own dissolution. Thus understood, Dialectic is discovered to be the life and soul of scientific progress, the dynamic which alone gives an immanent connexion and necessity to the subject-matter of science; and, in a word, is seen to constitute the real and true, as opposed to the external, exaltation above the finite." (Hegel, 2015, p. 126) Therefore, from *bad infinity* to *real infinity* is an inner transcendence, a dialectical process. *Real infinity* and *bad infinity* are the basic forms of infinity. Hegel put forward the profound dialectical conclusion that *real infinity* contains and subtracts *bad infinity*, and tries to grasp infinity concretely and realistically, and opposes abstract inferences about it. The transformation from *bad infinity* to *real infinity* is the transformation from understanding to reason, is a great leap in human understanding of infinity, and is the highest task of Hegel's philosophy.

The View of Dialectical Infinity holds that *real infinity* (or true infinity) is the connection and unity of the inner quality of *bad infinity*, so *real infinity* represents and reflects the quality of *bad infinity*; bad (or evil) infinity represents quantity of infinity (motion), *real infinity* represents quality (law, commonness or connection) of infinity. The View of Dialectical Infinity holds that the *actual infinity* separates the finite from the infinite, and looks at things from the point of view of stillness rather than movement, which has an internal irreconcilable contradiction, and is a metaphysical idealistic infinite

view. The concept of *actual infinity* is exactly the same thing as Transcendentalism, which in essence believes that the development of the world, the movement will have an end. However, Hegel's view of infinite is not only to see *the universal contact of the objective world (real infinity)*, but also the objective reality of "*the infinite process cannot be completed (bad infinity)*", so it is a scientific view of infinity, but also a view of dialectical infinity.

2.2 Brief of Hilbert's Thought of Infinite

In *On the Infinite*, Hilbert wrote: "In summary, let us return to our main theme and draw some conclusions from all our thinking about the infinite. Our principal result is that the infinite is nowhere to be found in reality. It neither exists in nature nor provides a legitimate basis for rational thought—a remarkable harmony between being and thought. In contrast to the earlier efforts of Frege and Dedekind, we are convinced that certain intuitive concepts and insights are necessary conditions of scientific knowledge, and logic alone is not sufficient. Operating with the infinite can be made certain only by the finitary." (Benacerraf & Putnam, 2003, p. 231); "The role that remains for the infinite to play is solely that of an idea—if one means by an idea, in Kant's terminology, a concept of reason which transcends all experience and which completes the concrete as a totality—that of an idea which we may unhesitatingly trust within the framework erected by our theory." (Benacerraf & Putnam, 2003, p. 231)

The above discussion fully illustrates Hilbert's idea of infinite: *Infinity* is not an objective existence, but infinite exists outside the experience as an ideal concept, and we can only master the infinite through finite.

2.3 On the Rationality of Existence of Actual Infinity, the View of Dialectical Infinity Is Consistent with Hilbert's Thought

In the view of dialectical infinite, infinite is an objective existence, such as time and space are natural objective infinite, and the unextinction of matter is also a kind of objective infinite, which confirms the rationality of the existence of an infinite set. At this point, the view of dialectical infinity is consistent with Hilbert thought, both confirming the rationality of the existence of *actual infinite*; although the former regards the infinite as an objective existence and the latter as an ideal existence.

However, the main difference between their ideas is on the view of "*whether the infinite process can be completed*", the view of dialectical infinity believes that the infinite process as the main body understanding object process, as the contradiction between finite and infinite, is impossible to complete, impossible to end, and Hilbert thinks that infinite process can be completed, end, abandoned the contradiction between finite and infinite, with subjective instead of objective, imposed the product of thinking on the objective material world, thus is a kind of idealistic epistemology. The existence of infinity and the incompleteness of process are two completely different concepts, they are two aspects of contradiction and irreplaceable; it is because of their existence that there are contradictions between finite and infinite. Therefore, the view of dialectical infinity does not oppose Cantor's theory of infinite set.

2.4 To Scientifically Grasp an Infinite, the View of Dialectical Infinity and Hilbert's Thought of Infinite Is Still Consistent in the General Direction

In the view of the viewer of dialectical infinite, the *bad infinite* is abstract, cannot end, cannot grasp, is the external expression of infinite, reflects the movement of infinite, but *real infinite* is concrete, realistic, is a finite, can be understood, grasp, it is the embodiment of the inner quality of infinite.

Hilbert believes that we cannot find infinite in the real world, infinite here means *bad infinite*, because the material world is infinite, and human thinking is limited, so any *bad infinite* is what our human intelligence cannot grasp—that is un-experienced, which deeply reflects the contradiction between the subject and object (i.e., the contradiction between finite and infinite). However, we human beings can understand the infinite world, this understanding means that we human beings can grasp the inner law of infinite through finite method, make the abstract *bad infinite* concrete and realistic, so as to reveal and grasp the inner essence of *bad infinite* (that is, *real infinite*, one finite, one law). For example, in the mathematical analysis to find a limit of a sequence of numbers, that is, through the finite method to master the essence of the infinite sequence—limit (the limit is a real infinite). Therefore, the thought of *real infinity* in the view of dialectical infinity is completely consistent with Hilbert's idea of mastering infinity through finiteness, except that Hilbert has failed to see the nature of the problem from a philosophical height to effectively distinguish between *bad and real infinity*. In fact, the history of scientific understanding since human civilization has all proved such an iron fact that human beings know the infinite by knowing *real infinite*. The *infinite* used in our theory are all regular infinite, and we humans can only master the infinite in the meaning of *real infinite*.

3. Summary

The formalism, represented by Hilbert, is contradictory in itself on *infinite problems*. On the one hand, they admit the theory of infinite set, thus they admit the reality of *infinite*, so they are typical theorists of *actual infinity*, but on the other hand, they insist on ***the principle of finitism*** in the concrete application, and worry about the use of the concept and method of *actual infinity*, almost as much as the intuitionists believe that credibility can only exist in the finite. They believe that infinity objects are beyond intuition and untrustworthy, so they are also theorists of *potential infinity*. So Mr. Zhu Wujia called him more vividly as an *theorists of actual infinity in the front* and a *theorists of finity behind* (Zhu, 2008, p. 146). This fully illustrates the contradiction of Hilbert on the problem of infinity, the fundamental reason is that he does not realize the dialectics of the problem of infinity, does not see that infinity is the contradictory unity of *real infinity* and *bad infinity*, and does not see that *the existence of infinity* and *the completeness of processes* are completely different concepts. On the existence of *infinity object*, the Formalism School is consistent with the Logicism School and opposite to the Intuitionism School. Hilbert insists on ***the principle of finitism*** in the concrete mathematical reasoning method, which stands on the same line as the Intuitionist School. Therefore, Hilbert is a less determined

theorist of *actual infinity*, a contradictory theorist of *actual infinity*, whose thought of infinite is closer to the view of dialectical infinite.

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