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Original Paper

List Edge Colorings of Planar Graphs with 7-cycles Containing at Most Two Chords

Wenwen Zhang[1*]

[1] School of Date and Computer Science, Shandong Women's University, Jinan, 250300, China

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Abstract

In this paper we prove that if G is a planar graph, and each 7-cycle contains at most two chords, then G is edge-k-choosable, where $k = \max\{8, \Delta(G) + 1\}$.

Keywords

List edge coloring, planar graph, cycle

1. Introduction

All graphs considered here are finite, simple and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For vertex $v \in V(G)$, let $E(v)$ be the set of edges incident with v. The degree of v in G, denoted $d(v)$, is the cardinality of $E(v)$. A *k-vertex*, k^-*-vertex* or k^+*-vertex* is a vertex of degree k, at most k or at least k, respectively. We denote the maximum degree of G by $\Delta(G)$ and minimum degree of G by $\delta(G)$. A k (or k^+)-vertex adjacent to a vertex x is called a k (or k^+)-neighbor of x. A k-cycle is a cycle of length k. Given a cycle C of length k in G, an edge $xy \in E(G)\backslash E(C)$ is called a *chord* of C if $x, y \in V(C)$. Such a cycle C is also called a chordal-k-cycle.

Let G be a plane graph, $F(G)$ be the face set of G. The degree of a face f, denoted by $dG(f)$ is the number of edges incident with f where each cut edge is counted twice. A k-, k^+-face is a face of degree k, at least k. A k-face of G is called an $(i_1, i_2, \dots, i_k)$-face if the vertices in its boundary are of degrees $i_1, i_2, \dots, i_k$ respectively. For a vertex $v \in V(G)$, we denote by $f_k(v)$ the number of k-faces incident with v.

A graph is *k-edge-colorable*, if its edges can be colored with k colors such that adjacent edges receive different colors. The *edge chromatic number* of a graph G, denoted by $\chi'(G)$, is the smallest integer k such that G is k-edge-colorable. We say that L is an edge assignment for G if it assigns a list $L(e)$ of colors to each edge e of G. If G has a proper edge-coloring φ such that $\varphi(e) \in L(e)$ for each edge e of G, then we say that G is edge-L-colorable and φ is an edge-L-coloring of G. The graph G is edge-k-choosable if it is edge-L-colorable for every edge assignment L satisfying $|L(e)| \geq k$ for each edge $e \in E(G)$, where k is a positive integer. The list-edge-chromatic-number $\chi'_L(G)$ of G is the smallest k such that G is edge-k-choosable.

List edge coloring was introduced by Vizing (Haggkvist & Chetwynd, 1992), later Bollobas and Harris (1985). They posed the following conjecture which is called the List Coloring Conjecture.

Conjecture 1. *For any multigraph G,* $\chi'_L(G) = \chi'(G)$.

Conjecture 1 was verified for some special classes of graphs, including bipartite multigraphs (Galvin, 1995), complete graphs of odd order (Haggkvist & Janssen, 1997), multicircuits (Woodall, 1999), graphs with $\Delta(G) \geq 12$ which can be embedded in a surface of non-negative characteristic (Borodin, Kostochka, & Woodall, 1997), and outer planar graphs (Wang & Lih, 2001). Vizing (see Kostochka, 1992) proposed a weaker conjecture as follows.

Conjecture 2. *Every graph G is edge-*$(\Delta(G) + 1)$*-choosable.*

Harris (n.d.) proved that $\chi'_L(G) \leq 2\Delta(G) - 2$ if G is a graph with $\Delta(G) \geq 3$. This implies Conjecture 2 for the case $\Delta(G) = 3$. Juvan et al. (1999) settled the case for $\Delta(G) = 4$. Conjecture 2 was verified for some special classes of graphs, including complete graphs (Haggkvist & Janssen, 1997), graphs with girth at least $8\Delta(\ln \Delta + 1.1)$ (Haggkvist & Chetwynd, 1992), planar graphs with $\Delta \geq 8$ (Bonamy, 2015). For planar graphs with some local conditions, see Hou, Liu and Cai (2009), Ma, Wang, Cai and Zhang (2011), Wang and Wu (2018).

Ca, Ge, Zhang and Liu (2011) proved that if G is a planar graph without chordal 7-cycles, then G is edge-k-choosable, where $k = \max\{8, \Delta(G) + 1\}$. In this paper, we will extend this result to planar graphs in which all 7-cycles contain at most two chords and get the following theorem.

Theorem 3. *Let G be a planar graph in which each* 7*-cycle contains at most two chords. Then G is edge-k-choosable, where* $k = \max\{8, \Delta(G) + 1\}$.

2. Structural Properties of Planar Graphs with 7-Cycles Containing at Most Two Chords

Lemma 4. *Let G be a planar graph in which each* 7*-cycle contains at most two chords. Then at least one of the following holds.*

(1) *G has an edge uv with* $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}$;

(2) *G has an even cycle* $C = v_1v_2...v_{2n}v_1$ *with* $d(v_1) = d(v_3) = ... = d(v_{2n-1}) = 3$.

Proof. Since every planar graph with maximum degree $\Delta(G) \geq 8$ has chromatic index $\Delta(G)+1$ (see Bonamy, 2015), we assume that $\Delta(G) \leq 7$ in the following proof. Suppose that G is a minimum counterexample to Lemma 4 in terms of the sums of the number of vertices and edges. It is obvious that G is connected. By the choice of G, we have there observations.

(a) By the assumption, for any edge uv, $d(u) + d(v) \geq \max\{10, \Delta(G) + 3\}$ since (1) does not hold. So $\delta(G) \geq 3$ and all 3-vertices must be adjacent to maximum degree vertices. Besides, any 4-vertex is only adjacent to vertices of degree at least $\Delta(G) - 1$.

(b) Since G contains no 7-cycles with three chords, so for any 6^+-vertex $v \in V(G)$, v is not incident with five 3-faces f_1, f_2, f_3, f_4, f_5 such that f_i and $f_i + 1$ are adjacent for all $i = 1, 2, 3, 4$. Then $f3(v) \leq [\frac{4}{5}d(v)]$.

(c) Let G_3 be the subgraph induced by the edges incident with all 3-vertices of G. Then G_3 is a forest and it contains a bipartite subgraph $G' = (V_1, V_2)$ with two partite sets V_1 and V_2, such that $d_{G}'(v) = 2$ for each vertex $v \in V_1$ and $d_{G}'(v) = 1$ for each vertex $v \in V_2$. If $uv \in G'$ and $d_G(u) = 3$, then v is called a 3-master of u and u is called a dependent of v. Note that every 3-vertex has exactly two 3-masters and each 7^+-vertex can be the 3-master of at most one 3-vertex.

Next we show that (c) is true. By (a), any two 3-vertices are not adjacent, that is, G_3 does not contain odd cycles. Thus G_3 is a bipartite graph with partite sets V_1, V_2, so that $V(G) = V_1 \cup V_2$ and for each vertex $v \in V_1$, $d_G(v) = 3$; for each vertex $v \in V_2$, $d_G(v) = \Delta$. Since G does not satisfy (2), G_3 contains no even cycles. So G_3 is a forest. For any component of G_3, we select a vertex u with $d_G(u) = 3$ as the root of the tree. Thus, every 3-vertex has exactly two children. We obtain G' by letting $V_2 = \{v : v$ is a child of a 3-vertex$\}$ and $E(G') = \{uv : u$ is 3-vertex and v is a child of $u\}$. So (c) holds.

Since G has properties (a), and G contains no 7-cycles with three chords. Suppose that v is a 5-vertex in G. Then we can get the following observations easily:

(O1) If $f_3(v) = 4$ and $f_4(v) = 1$ (as in Figure 1), then $f_5+(v_1) \geq 2$;

(O2) If $f_3(v) = 5$ and $f_4(v) = 1$, then for any neighbor x of v, $f_5+(x) \geq 2$.

Suppose that G is embedded in the plane. By Euler's formula $|V(G)|-|E(G)|+|F(G)| = 2$, we have

$$\sum_{x \in V(G)} (d(x) - 4) + \sum_{x \in F(G)} (d(x) - 4) = -8 < 0$$

We define ch to be the initial charge. Let $ch(x) = d(x) - 4$ for each $x \in V \cup F$. So $\sum_{x \in V \cup F} ch(x) < 0$. Then we apply the following rules to redistribute the initial charge that leads to a new charge $ch'(x)$ to each $x \in V \cup F$. Since our rules only move charges around, and do not affect the sum. If we can show that $ch'(x) \geq 0$ for each x, then we get an obvious contradiction, $0 \leq \sum_{x \in V \cup F} ch'(x) = \sum_{x \in V \cup F} ch(x) < 0$. which completes our proof.

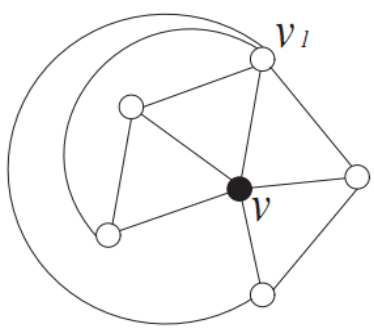

Figure 1. Black Vertices do not Have Neighbors Other than Presented in the Picture, White Vertices can be Adjacent to Some Other Vertices

The discharging rules are defined as follows.

R1. Every 3-vertex receives $\frac{1}{2}$ from each of its two 3-masters.

R2. Let f be a 3-face uvw and assume that $d(u) \leq d(v) \leq d(w)$.

R2.1 If $d(u) = 3$ or 4, then f receives $\frac{1}{2}$ from v and w respectively;

R2.2 If $d(u) \geq 5$, then f receives $\frac{1}{3}$ from u, v and w respectively.

R3. Let f be a 5^+-face and t the number of 5-vertices satisfying $f_3(v) = 4$ on f.

R3.1 If $t = 0$, then every of vertices incident with f receives $\frac{d(f)-4}{d(f)}$ from f;

R3.2 Otherwise $t \geq 1$. Suppose v is such a vertex, then the every of remaining vertices incident with f receives $\frac{d(f)-4}{d(f)-2}$ from f besides its two neighbors on f.

R4. Let v be a 5-vertex.

R4.1 If $f_3(v) = 4$ and $f_4(v) = 1$ (as in Figure 1), then v receives at least $\frac{1}{3}$ from w by (O1);

R4.2 If $f_3(v) = 5$, then v receives $\frac{1}{5}$ from each of the neighbors by (O2).

Now, let's begin to check $ch'(x) \geq 0$ for all $x \in V \cup F$. Let $f \in F(G)$. Then $d(f) \geq 3$. If $d(f) = 3$, then $ch'(f) = ch(f)+\min\{2\times\frac{1}{2}, 3\times\frac{1}{3}\} = 0$ by R2. If $d(f) = 4$, then $ch'(f) = ch(f) = 0$. If $d(f) \geq 5$, then $ch'(f) \geq ch(f)-\frac{d(f)-4}{d(f)}\times d(f) = 0$ or $ch'(f) \geq ch(f)-\frac{d(f)-4}{d(f)-2}\times(d(f)-2) = 0$ by R3.

Let $v \in V(G)$. Then $d(v) \geq 3$. If $d(v) = 3$, then v is exactly adjacent to two 3-masters, so $ch'(v) = ch(v) + 2 \times \frac{1}{2} = 0$ by R1. If $d(v) = 4$, then $ch'(v) \geq 0 + \min\{0, \frac{d(f)-4}{d(f)-2}, \frac{d(f)-4}{d(f)}\} = 0$ by R3.

In the following we check the cases that $d(v) = 5, 6, 7$.

Case 1. Let v be a 5-vertex. Then $ch(v) = 1$ and all neighbors of 5-vertex should be 5^+-vertices by (a). If $f_3(v) = 5$, then v receives $\frac{1}{5}$ by R4.2. So $ch'(v) \geq 1+5\times\frac{1}{5}-5\times\frac{1}{3} = \frac{1}{3} > 0$ by R2. Suppose that $f_3(v) = 4$. If the remaining face is a 4-face (as Figure 1), then $ch'(v) \geq 1 - 4\times\frac{1}{3} + \frac{1}{3} = 0$ by R2 and R4.1; otherwise the remaining face is a 5^+-face. Then v receives $\frac{d(f)-4}{d(f)-2} \geq \frac{1}{3}$ from the 5^+-face by R3.2. So

$ch'(v) \geq 1 - 4\times\frac{1}{3} + \frac{d(f)-4}{d(f)-2} \geq 0$ by R2 and R3.2.

If $f_3(v) \leq 3$, then v may send some charge to its 5-neighbors. So there are two subcases.

Subcase 2.1 v sends no charge to some 5-neighbor.

Then $ch'(v) \geq 1 - 3\times\frac{1}{3} = 0$ by R2.

Subcase 2.2 v sends some charge to some 5-neighbor.

Suppose that v is adjacent to a 5-vertex w such that $f_3(w) = 4$ and $f_4(w) = 1$ (as in Figure 1), then $f_3(v) \leq 2$, $f_4(v) = 1$ and $f_{5^+}(v) = 2$. So $ch'(v) \geq 1 - 2\times\frac{1}{3} - \frac{1}{3} = 0$ by R2 and R4.1.

Suppose that v is adjacent to a 5-vertex w such that $f_3(w) = 5$. Then $f_3(v) \leq 3$, $f_{5^+}(v) \geq 2$ and each

5^+-face sends at least $\frac{1}{5}$ to v by R3. So $ch'(v) \geq 1 + 2 \times\frac{1}{5} - 3 \times\frac{1}{3} - \frac{1}{5} = \frac{1}{5} > 0$ by R2 and R4.2.

Case 2. Let v be a 6-vertex. Then $ch(v) = 2$, $f_3(v) \leq 4$ by (b).

If v is not adjacent to a 5-vertex w such that $f_3(w) = 4$ and $f_4(w) = 1$ or $f_3(w) = 5$, then v sends no charge to its 5-neighbors, so $ch'(v) \geq 2 - 4 \times \max\{\frac{1}{3}, \frac{1}{2}\} = 0$ by R3.

Suppose that v is adjacent to a 5-vertex w such that $f_3(w) = 4$ and $f_4(w) = 1$ (as in Figure 1). Note that v is adjacent to only one such vertex w because each 7-cycle contains at most two chords in G. Then $f_3(v) \leq 3$, $f_4(v) = 1$ and $f_5+(v) = 2$. So $ch'(v) \geq 2 - 3 \times\frac{1}{2} - \frac{1}{3} = \frac{1}{6} > 0$ by R2 and R4.1.

Suppose that v is adjacent to a 5-vertex w such that $f_3(w) = 5$. Note that v may be adjacent to two such vertices. Then $f_3(v) \leq 4$ and $f_5+(v) \geq 2$. So $ch'(v) \geq 2+2\times\frac{1}{5} - 2\times\frac{1}{3} - 2\times\frac{1}{2} - 2\times\frac{1}{3} = \frac{1}{15} > 0$ by R2, R3.1 and R4.2.

Case 3. Let v be a 7-vertex. Then $ch(v) = 3$, $f_3(v) \leq 5$ by (b).

If v is not adjacent to a 5-vertex w such that $f_3(w) = 4$ and $f_4(w) = 1$ or $f_3(w) = 5$, then v sends no charge to its 5-neighbors, so $ch'(v) \geq 3 - 5 \times \max\{\frac{1}{3}, \frac{1}{2}\} - \frac{1}{2} = 0$ by R1 and R2.

Suppose that v is adjacent to a 5-vertex w such that $f_3(w) = 4$ and $f_4(w) = 1$ (as in Figure 1). Now $f_3(v) \leq 4$ and v is adjacent to at most two such vertices because each 7-cycle contains at most two chords in G. So $ch'(v) \geq 3 - 2 \times\frac{1}{2} - 2 \times\frac{1}{3} - \frac{1}{2} - 2 \times\frac{1}{3} = \frac{1}{6} > 0$ by R1, R2 and R4.1.

Suppose that v is adjacent to a 5-vertex w such that $f_3(w) = 5$. Note that v may be adjacent to two such vertices. Then $f_3(v) \leq 5$ and $f_5+(v) \geq 2$. So $ch'(v) \geq 3 + 2 \times\frac{1}{5} - 2 \times\frac{1}{3} - 3 \times\frac{1}{2} - \frac{1}{2} - 2 \times\frac{1}{3} = \frac{1}{15} > 0$ by R1, R2, R3.1 and R4.2.

3. Proof of Theorem 3

Proof. The proof is carried out by contradiction. Suppose that G is a counterexample to our theorem with the minimum number of edges and G is any planar graph in which every 7-cycle contains at most two chords. Then there is an edge assignment L with $|L(e)| \geq k$ for all $e \in E(G)$, where $k = \max\{8, \Delta(G) + 1\}$, such that G is not edge-L-colorable. By Lemma 4, we consider two cases as follows.

Case 1. G contains an edge uv with $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}$. Consider the graph $G' = G - uv$. By inductive hypothesis, G has an edge-L- coloring φ, where L is an edge assignment with $|L(e)| \geq k$ for all $e \in E(G')$ and $k = \max\{8, \Delta(G') + 1\}$. Since there exist at most $\max\{7, \Delta(G)\}$ edges adjacent in G to uv and $|L(uv)| \geq \max\{8, \Delta(G) + 1\}$, we can color uv with some color from $L(uv)$ that was not used by φ on the edges adjacent to uv. It is easy to see that the resulting coloring is an edge-L-coloring of G.

Case 2. G contains an even cycle $c = v_1v_2...v_{2n}v_1$ with $d(v_1) = d(v_3) = ... = d(v_{2n}-1) = 3$. Let G' be the subgraph of G obtained by deleting the edges of C. By inductive hypothesis, G' has an edge-L-coloring φ, where L is an edge assignment with $|L(e)| \geq k$ for all $e \in E(G)$ and $k = \max\{8, \Delta(G) + 1\}$. Define a new edge assignment $L'(e)$ of C such that $L'(e) = L(e) \setminus \{\varphi(e')|e' \in E(G')$ is adjacent to e in $G\}$ for each $e \in E(C)$. It is easy to see that $|L'(e)| \geq 2$ for each $e \in E(C)$. It follows from Erdős et al. (1979) that an even cycle is edge-2-choosable (since an even cycle is also a bipartite graph). So C is edge-L-colorable and it follows that G is edge-L-colorable. This completes the proof of Theorem 3.

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