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Original Paper

Reducible Relationships between Generalized Syllogisms with the Quantifiers in Square{*at least half of the*}

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Abstract
The main conclusions of this paper are as follows: Theorem 1 proves the validity of the generalized syllogism SAI-3. Theorem 2 takes the syllogism SAI-3 as the fundamental axiom, and then deduces the other 14 non-trivial valid generalized syllogisms with the quantifiers in Square{at least half of the}. It means that there are reducible relationships between these 15 syllogisms, and that the above knowledge mining processes are consistent. This innovative achievement can promote in-depth research on generalized syllogistic and provide methodological inspiration for knowledge mining in artificial intelligence.
Keywords
generalized syllogisms, validity, reducible relationships, knowledge mining

1. Introduction

In natural language reasoning, in addition to common classical syllogism reasoning (Łukasiewicz, 1957), there is generalized syllogism reasoning (Hao, 2024a-b). A non-trivial generalized syllogism contains at least one non-trivial generalized quantifier (Peters & Westerståhl, 2006; Xu & Yu, 2024). There are only four classical quantifiers as follows: *some*, *no*, *not all,* and *all*, and they form Square{*some*} (Wang & Yuan, 2024). Non-trivial generalized quantifiers are non-classical ones, such as *at least half of the, fewer than half of the, at most half of the, most.* They form Square{*at least half of the*} (Ma & Cao, 2024). Although there has been some works about generalized syllogisms (Moss, 2010; Endrullis & Moss, 2015; Cao & Li, 2024), there are still many interesting questions to be studied. This paper only studies the validity and reducibility of the generalized syllogisms involving the eight quantifiers from Square{*some*} and Square{*at least half of the*}.

2. Preliminaries

In this paper, let *p*, *t*, and *n* be variables, and *P, T*, and *N* be the sets that composed of *p, t, n,* respectively. *D* be the domain of variables. Let λ, ω, ψ, and δ be well-formed formulas (shorted as wff). '$|P\cap N|$' represents the cardinality of the intersection of the set *P* and *N,* '⊢ω' says that ω is provable, and 'ω$=_{def}$

δ' that ω can be defined by δ. The operators (such as ¬, →, ∧, ↔) are basic ones in set theory (Halmos, 1974).

Let *Q* is a quantifier, ¬*Q*, *Q*¬, and ¬*Q*¬ are respectively its outer, inner, dual negative quantifier. These four quantifiers form a modern Square{*Q*}={*Q*, ¬*Q*, *Q*¬, ¬*Q*¬} (Hao, 2024b). For example, Square{*some*}={*some*, *no*, *not all, all*}, and Square{*at least half of the*}={*at least half of the, fewer than half of the, at most half of the, most*}. The propositions containing these 8 quantifiers from Square{*some*} and Square{*at least half of the*} are respectively called Proposition *I*, *E*, *O*, *A*, *S*, *F*, *H*, and *M* (Ma & Cao, 2024). This paper only studies the non-trivial generalized syllogisms composed of these 8 propositions, and these syllogisms include at least one of the last four propositions. Thus, '*at least half of the(t, n)∧all(t, p)→some(p, n)*' is a third figure syllogism, which can be abbreviated as *SAI-3*.

Example 1:

Major premise: At least half of my flowers are roses.

Minor premise: All flowers are plants.

Conclusion: Some plants are roses.

Let *t*, *n*, and *p* be variables that represent a flower, a rose, and a plant, respectively, then this syllogism can be symbolized as '*at least half of the(t, n)∧all(t, p)→some(p, n)*'.

Special note: It was assumed that the set composed of the subject variable for a categorical proposition containing one of the four quantifiers (*all*, *not all, most*, *and at most half of the*) is not an empty set.

3. Generalized Syllogism System with the quantifiers '*at least half of the*'

This system consists of the following parts:

3.1 Primitive Symbols

(1) variables: *p, t, n*

(2) quantifiers: *some, at least half of the*

(3) operators: ¬, →

(4) brackets: (,)

3.2 Formation Rules

(1) If *Q* is a quantifier, *p* and *n* are variables, then *Q(p, n)* is a wff.

(2) If λ and ω are wffs, then so are ¬λ and λ→ω.

(3) Only the sentences formed on the basis of the above two rules are wffs.

3.3 Basic Axioms

A1: If λ is a valid proposition in classical logic, then ⊢λ.

A2: ⊢*at least half of the(t, n)∧all(t, p)→some(p, n)* (i.e. the syllogism *SAI-3*).

3.4 Rules of Deduction

Rule 1 (Antecedent strengthening): If ⊢(λ∧ω→ψ) and ⊢(δ→λ), then ⊢(δ∧ω→ψ).

Rule 2 (subsequent weakening): If ⊢(λ∧ω→ψ) and ⊢(ψ→δ), then ⊢(λ∧ω→δ).

Rule 3 (anti-syllogism): If ⊢(λ∧ω→ψ), then ⊢(¬ψ∧λ→¬ω).

3.5 Relevant Definitions

D1: $(\omega\wedge\delta)=_{def}\neg(\omega\rightarrow\neg\delta)$;

D2: $(\omega\leftrightarrow\delta)=_{def}(\omega\rightarrow\delta)\wedge(\delta\rightarrow\omega)$;

D3: $(Q\neg)(p, n)=_{def} Q(p, D-n)$;

D4: $(\neg Q)(p, n)=_{def}$ It is not the case that $Q(p, n)$;

D5: *some(p, n)* is true iff $P\cap N\neq\varnothing$ is true;

D6: *no(p, n)* is true iff $P\cap N=\varnothing$ is true;

D7: *not all(p, n)* is true iff $P\not\subseteq N$ is true;

D8: *all(p, n)* is true iff $P\subseteq N$ is true;

D9: *at least half of the(p, n)* is true iff $|P\cap N|\geq 0.5|P|$ is true;

D10: *fewer than half of the(p, n)* is true iff $|P\cap N|<0.5|P|$ is true;

D11: *at most half of the(p, n)* is true iff $|P\cap N|\leq 0.5|P|$;

D12: *most(p, n)* is true iff $|P\cap N|>0.5|P|$ is true.

3.6 Relevant Facts

Fact 1 (inner negation):

(1.1) ⊢*all(p, n)*↔*no*¬*(p, n)*;

(1.2) ⊢*no(p, n)*↔*all*¬*(p, n)*;

(1.3) ⊢*some(p, n)*↔*not all*¬*(p, n)*;

(1.4) ⊢*not all(p, n)*↔*some*¬*(p, n)*;

(1.5) ⊢*most(p, n)*↔*fewer than half of the*¬*(p, n)*;

(1.6) ⊢*fewer than half of the(p, n)*↔*most*¬*(p, n)*;

(1.7) ⊢*at least half of the(p, n)*↔*at most half of the*¬*(p, n)*;

(1.8) ⊢*at most half of the(p, n)*↔*at least half of the*¬*(p, n)*.

Fact 2 (outer negation):

(2.1) ⊢¬*all(p, n)*↔*not all(p, n)*;

(2.2) ⊢¬*not all(p, n)*↔*all(p, n)*;

(2.3) ⊢¬*no(p, n)*↔*some(p, n)*;

(2.4) ⊢¬*some(p, n)*↔*no(p, n)*;

(2.5) ⊢¬*most(p, n)*↔*at most half of the(p, n)*;

(2.6) ⊢¬*at most half of the(p, n)*↔*most(p, n)*;

(2.7) ⊢¬*fewer than half of the(p, n)*↔*at least half of the(p, n)*;

(2.8) ⊢¬*at least half of the(p, n)*↔*fewer than half of the(p, n).*

Fact 3 (symmetry):

(3.1) ⊢*some(p, n)*↔*some(n, p)*;

(3.2) ⊢*no(p, n)*↔*no(n, p)*.

Fact 4 (subordination) :

(4.1) ⊢*all(p, n)*→*some(p, n)*;

(4.2) ⊢*no(p, n)*→*not all(p, n)*;

(4.3) ⊢*all(p, n)*→*most(p, n)*;

(4.4) ⊢*most(p, n)*→*some(p, n)*;

(4.5) ⊢*all(p, n)*→*at least half of the(p, n)*;

(4.6) ⊢*at least half of the(p, n)*→*some(p, n)*;

(4.7) ⊢*at least half of the(p, n)*→*most(p, n)*;

(4.8) ⊢*at most half of the(p, n)*→*fewer than half of the(p, n)*;

(4.9) ⊢*fewer than half of the(p, n)*→*not all(p, n)*;

(4.10) ⊢*at most half of the(p, n)*→*not all(p, n)*;

(4.11) ⊢*no(p, n)*→*fewer than half of the(p, n)*;

(4.12) ⊢*no(p, n)*→*at most half of the(p, n)*.

The above facts are the basis facts in generalized quantifier theory (Peters & Westerståhl, 2006).

4. Knowledge Mining about Valid Generalized Syllogisms

The following Theorem 1 proves the validity of the generalized syllogism *SAI-3*. Taking the syllogism *SAI-3* as the fundamental axiom in Theorem 2, the other 14 valid generalized syllogisms can be derived by the above facts and definitions.

Theorem 1 (*SAI-3*): The generalized syllogism *at least half of the(t, n)*∧*all(t, p)*→*some(p, n)* is valid.

Proof: Suppose that *at least half of the(t, n)* and *all(t, p)* are true, thus $|T \cap N| \geq 0.5|T|$ is true according to Definition D9, and $T \subseteq P$ is true with the help of Definition D8. It follows that $P \cap N \neq \varnothing$. This can be proven using the method of contradiction. Suppose that $P \cap N = \varnothing$, thus $T \cap N = \varnothing$ in line with $T \subseteq P$. This contradicts $|T \cap N| \geq 0.5|T|$. Therefore, $P \cap N \neq \varnothing$ holds. It can be concluded that *some(p, n)* is true in the light of Definition D5, just as desired.

Theorem 2: There are 14 valid generalized syllogisms inferred from *SAI-3*:

(1) ⊢*SAI-3*→*ASI-3*

(2) ⊢*SAI-3*→*HAO-3*

(3) ⊢*SAI-3*→*EAF-1*

(4) ⊢*SAI-3*→*EAF-1*→*EAF-2*

(5) ⊢*SAI-3*→*ESO-2*

(6) ⊢*SAI-3*→*ESO-2*→*ESO-1*

(7) ⊢*SAI-3*→*ASI-3*→*ESO-3*

(8) ⊢*SAI-3*→*ASI-3*→*ESO-3*→*ESO-4*

(9) ⊢*SAI-3*→*EAF-1*→*AAM-1*

(10) ⊢*SAI-3*→*EAF-1*→*EAF-2*→*AEF-2*

(11) ⊢*SAI-3*→*EAF-1*→*EAF-2*→*AEF-2*→*AEF-4*

(12) ⊢*SAI-3*→*ESO-2*→*AHO-2*

(13) ⊢*SAI-3*→*ESO-2*→*ESO-1*→*ASI-1*

(14) ⊢*SAI-3*→*ESO-2*→*ESO-1*→*ASI-1*→*SAI-4*

Proof:

[1] ⊢*at least half of the(t, n)*∧*all*(*t, p*)→*some(p, n*) (i.e. *SAI-3*, basic axiom A2)

[2] ⊢*at least half of the(t, n)*∧*all*(*t, p*)→*some(n, p)* (i.e. *ASI-3*, by [1] and Fact (3.1))

[3] ⊢*at most half of the (t, n)*∧*all*(*t, p*)→*not all*¬*(p, n)* (by [1], Fact (1.7) and (1.3))

[4] ⊢*at most half of the*(*t, D–n*)∧*all*(*t, p*)→*not all*(*p, D–n*) (i.e. *HAO-3*, by [1] and Definition D3)

[5] ⊢¬*some*(*p, n*)∧*all*(*t, p*)→¬*at least half of the(t, n)* (by [1] and Rule 3)

[6] ⊢*no(p, n)*∧*all*(*t, p*)→*fewer than half of the(t, n)* (i.e. *EAF-1*, by [5], Fact (2.4) and (2.8))

[7] ⊢*no(n, p)*∧*all*(*t, p*)→*fewer than half of the(t, n)* (i.e. *EAF-2*, by [6] and Fact (3.2))

[8] ⊢¬*some(p, n)*∧*at least half of the(t, n)*→¬*all*(*t, p*) (by [1] and Rule 3)

[9] ⊢*no(p, n)*∧*at least half of the(t, n)*→*not all*(*t, p*) (i.e. *ESO-2*, by [8], Fact (2.4) and (2.1))

[10] ⊢*no(n, p)*∧*at least half of the(t, n)*→*not all*(*t, p*) (i.e. *ESO-1*, by [9] and Fact (3.2))

[11] ⊢*at least half of the(t, n)*∧*no*¬(*t, p*)→*not all*¬*(n, p)* (by [2], Fact (1.1) and (1.3))

[12] ⊢*at least half of the(t, n)*∧*no*(*t, D–p*)→*not all*(*n, D–p*) (i.e. *ESO-3*, by [11] and Definition D3)

[13] ⊢*at least half of the(t, n)*∧*no*(*D–p, t*)→*not all*(*n, D–p*) (i.e. *ESO-4*, by [12] and Fact (3.2))

[14] ⊢*all*¬*(p, n)*∧*all*(*t, p*)→*most*¬*(t, n)* (by [6], Fact (1.2) and (1.6))

[15] ⊢*all*(*p, D–n*)∧*all*(*t, p*)→*most*(*t, D–n*) (i.e. *AAM-1*, by [14] and Definition D3)

[16] ⊢*all*¬*(n, p)*∧*no*¬(*t, p*)→*fewer than half of the(t, n)* (by [7], Fact (1.1) and (1.2))

[17] ⊢*all(n, D–p)*∧*no*(*t, D–p*)→*fewer than half of the(t, n)* (i.e. *AEF-2*, by [16] and Definition D3)

[18] ⊢*all(n, D–p)*∧*no*(*D–p, t*)→*fewer than half of the(t, n)* (i.e. *AEF-4*, by [17] and Fact (3.2))

[19] ⊢*all*¬*(p, n)*∧*at most half of the*¬*(t, n)*→*not all*(*t, p*) (by [9], Fact (1.2) and (1.7))

[20] ⊢*all(p, D–n)*∧*at most half of the(t, D–n)*→*not all*(*t, p*) (i.e. *AHO-2*, by [19] and Definition D3)

[21] ⊢*all*¬*(n, p)*∧*at least half of the(t, n)*→*some*¬(*t, p*) (by [10], Fact (1.2) and (1.4))

[22] ⊢*all(n, D–p)*∧*at least half of the(t, n)*→*some*(*t, D–p*) (i.e. *ASI-1*, by [21] and Definition D3)

[23] ⊢*all(n, D–p)*∧*at least half of the(t, n)*→*some*(*D–p, t*) (i.e. *SAI-4*, by [22] and Fact (3.1))

The above proof processes are logical deduction ones. Therefore, the above knowledge mining processes are consistent. Theorem 2 demonstrates that the validity of the above 14 non-trivial generalized syllogisms can be derived from the validity of the syllogism *SAI-3*. In other words, there are reducible relationships between these 15 syllogisms with the quantifiers in Square{*at least half of the*}.

5. Conclusion

The main conclusions of this paper are as follows: Theorem 1 proves the validity of the generalized syllogism *SAI-3*. Theorem 2 takes the syllogism *SAI-3* as the fundamental axiom, and then deduces the other 14 non-trivial valid generalized syllogisms with the quantifiers in Square{*at least half of the*}. It means that there are reducible relationships between these 15 syllogisms, and that the above knowledge mining processes are consistent.

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