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*Original Paper*

# Positive Semigroups Using Resolvent Estimate Bounds on Sharp Growth Rates

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***Abstract***

*In this paper, we study growth rates for strongly continuous semigroups. We fixate that a growth rate for the resolvent estimate on imaginary lines implies a corresponding growth rate for the semigroup if either the underlying space is a Hilbert space, or the semigroup is asymptotically analytic, or if the semigroupis positive and the underlying space is an* $L_{(1+\epsilon)}$*-space or a space of continuous functions. Also proved variations of the main results on fractional domains; these are valid on more general Banach spaces by Jan Rozendaal and Mark Veraar. In the second part apply the main theorem to prove optimality in a classical example of a perturbed wave equation which shows unusual sequence of spectral behavior.*

***Keywords***

$C_0$*-semigroup, Polynomail growth, Positive semigroup, Fourier multiplier, Kreiss condition, Perturrbed wave equation*

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## 1. Introduction

Let $-A_j$ be the sequence of generators of a $C_0$-semigroup $(T^j((1+\epsilon)_{1+\epsilon}))_{\epsilon\geq -1}$ on a Banach space $X$. It can be quite difficult to verify the assumptions of the Hille$-Y$osida theorem to determine whether $(T^j(1+\epsilon))_{\epsilon\geq -1}$ are uniformly bounded, given that bounds for all powers of the resolvent of $A$are required. Hence it is of interest to determine spectral conditions that are easier to check and which imply specific growth behavior of $(T^j(1+\epsilon))_{\epsilon\geq -1}$, such as for example polynomial growth. One such condition: $\sigma(A_j)\subseteq\overline{\mathbb{C}_+}$ and for some $\epsilon\geq -1$.

$$\left\|\sum_j\left(\lambda_j+A_j\right)^{-1}\right\|\leq\frac{(1+\epsilon)}{Re\sum_j(\lambda_j)}\left(\lambda_j\in\mathbb{C}_+\right) \tag{1}$$

It is known that (1) implies $\left\|\sum_j T^j\,(1+\epsilon)\right\|\leq en(1+\epsilon)$ if $X$ is $n$-dimensional. Furthermore, as was shown in Jan and Mark (2018), if $X$ is a Hilbert space and (1) holds then $\left\|T^j(1+\epsilon)\right\|$ grows at most linearly in $(1+\epsilon)$, while there exist semigroups on general Banach spaces which satisfy (1) but grow exponentially.

There are many interesting strongly continuous semigroups with a polynomial growth rate. One important class is given by certain Schrödinger semigroups on $L^{(1+\varepsilon)}$-spaces,

$0\leq\epsilon\leq\infty$ that have sequence generator $\Delta+V$ for $V$an potential (Grigor'yan, 2006). Other examples arise from wave equations (Paunonen, 2014), delay equations (Sklyar & Polak, 2017), and the sequence operators matrices and multiplication sequence operators (Rozendaal & Veraar, 2017). In Arendt, Batty, Hieber and Neubrander (2011), Davies (2005), Eisner (2010) one may find additional examples of semigroups with interesting growth behavior.

The following is the main result of this article. It enables one to derive polynomial growth bounds for a semigroup from resolvent estimates similar to (1). Note that each eventually differentiable $C_0$-semigroup, and in specific all analytic semigroup, is asymptotically analytic. Also, condition (4) is satisfied if, e.g., $X=C_{u^j(1+\epsilon)}(\Omega)$ for $\Omega$ a metric space, or $X=C_0(\Omega)$ for $\Omega$ a locally compact space.

**Theorem 1.1.** Let $-A_j$ be the sequence of generators of a $C_0$-semigroup $(T^j(1+\epsilon))_{\epsilon\geq -1}$ on a Banach space $X$ such that $\mathbb{C}_-\subseteq\rho(A_j)$. Assume that one of the following conditions holds:

(1) $X$ is a Hilbert space;

(2) $(T^j(1+\epsilon))_{\epsilon\geq -1}$ is asymptotically analytic semigroup;

(3) $X=L^{(1+\varepsilon)}(\Omega)$ for $0\leq\epsilon<\infty$ and a measure space, and $T^j(1+\epsilon)$ is positive sequence operators for all $\epsilon\geq -1$.

(4) $X$ is a closed subspace of $C_{(1+\varepsilon)}(\Omega)$, for $\Omega$ a topological space, such that either $1_\Omega\in X$ or $X$is a sublattice, and $T^j(1+\epsilon)$ are positive operators for all $\epsilon\geq -1$.

If there exist $0\leq\epsilon<\infty$ and $\epsilon\geq 0$ such that

$$\left\|\sum_j(\lambda_j+A_j)^{-1}\right\|_{\mathcal{L}(X)}\leq(1+\epsilon)\sum_j\left(Re\left(\lambda_j\right)^{-(\beta+\epsilon)}+1\right)\left(\lambda_j\in\mathbb{C}_+\right), \tag{2}$$

then there exists $\epsilon\geq -1$ such that

$$\left\|\sum_j T^j(1+\epsilon)\right\|_{\mathcal{L}(X)} \le (\epsilon^2+2\epsilon+1)\left((1+\epsilon)^{(\beta+\epsilon)}+1\right)(\epsilon \ge -1). \tag{3}$$

In certainty, in the main text allow an arbitrary growth rate g in (2) and (3). It follows from Example 3.5 below that, for $\beta+\epsilon \in \mathbb{N}$, Theorem 1.1 is optimal up to arbitrarily small polynomial loss in (3).

For $\epsilon=-\beta$ and $X$ a Hilbert space, Theorem 1.1 reduces to the Gearhart-Pruss theorem, while for $\epsilon=-\beta$ and $(T^j(1+\epsilon))_{\epsilon\ge -1}$ a positive semigroup on an $L_{(1+\epsilon)}$-space one recovers a result.

For $0<\epsilon<\infty$ the inequality $\|R(\lambda_j,A_j)\| \ge dist(\lambda_j,\sigma(A_j))$ for $\lambda_j \in \rho(A_j)$ shows that $\overline{\mathbb{C}_-} \subseteq \rho(A_j)$, and then one can use a Neumann series argument to reduce to the casewhere $\epsilon=-\beta$.

For $\epsilon \ge -\beta$ it was previously known from Eisner and Zwart (2007) that (2) implies

$$\left\|\sum_j T^j\,(1+\epsilon)\right\|_{\mathcal{L}(X)} \le \epsilon^2+2\epsilon+1\left((1+\epsilon)^{2(\beta+\epsilon)-1}+1\right)(\epsilon \ge -1) \tag{4}$$

Whenever $(T^j(1+\epsilon))_{\epsilon\ge -1}$ has a so-called $(1+\varepsilon)$-integrable resolvent for some $0<\epsilon<\infty$. This property is satisfied by e.g. all $C_0$-semigroups on Hilbert spaces and analytic semigroups on general Banach spaces. If $\beta+\epsilon=1$ then (3) and (4) yield the same conclusion. In all other cases (3) improves (4). Theorem 1.1 also seems to be the first result of its kind for asymptotically analytic semigroups and for positive semigroups on $L^{1+\epsilon}$-spaces and spaces of continuous functions. Generation theorems for (semi)groups with polynomial growth were discussed. In contrast to these articles assume a priori that the relevant semigroup exists. Other results on semigroups of polynomial growth can be found in Boukdir (2015). Versions of Theorem 1.1 for Césaro type averages have been considered in Li, Sato and Shaw (2008), where also numerous counter examples are presented.

It was known from Eisner and Zwart (2007) that on general Banach spaces (3) implies

$$\left\|\sum_j (\lambda_j+A_j)^{-1}\right\|_{\mathcal{L}(X)} \le \frac{1+\epsilon}{\epsilon}\sum_j \left(Re(\lambda_j)^{-(\beta+\epsilon)-1}+1\right)(\lambda_j \in \mathbb{C}_+)$$

for some $\frac{1+\epsilon}{\epsilon} \ge 0$, thus providing a partial converse to Theorem 1.1. In Theorem 3.11 and Corollary 3.13 we extend this result and obtain a full characterization of polynomial stability of a semigroup in terms of properties of the resolvent of its sequence generators.

Also derive versions of Theorem 1.1 on fractional domains, where make other geometric assumptions on $X$. In specific, it is shown in Proposition 3.1 that on a general Banach space $X(1)$ implies at most linear growth for semigroup orbits with sufficiently smooth initial values. Also point out that, by choosing $\epsilon=-\beta$ and using a scaling argument, Theorem 1.1 and other results in Section 3imply various theorems about exponential stability.

Note here that the main result was applied to Schrödinger semigroups in Faupin and Fröhlich (2017), Theorem 5.4, to deduce cubic growth of the semigroup, whereas Theorem 1.1 immediately yields quadratic growth.

To prove Theorem 1.1 use the connection between stability theory and Fourier multipliers and which was renewed in Rozendaal and Veraar (2017), following the development of a theory of operator-valued $(L^{1+\varepsilon}, L^{1+\epsilon})$ Fourier multipliersin (Rozendaal & Veraar, 2017, 2018). In particular, Theorem 3.2 gives a Fourier multiplier criterion for abound as in (3) to hold, and Corollary 3.13 gives a characterization of polynomial growth and uniform boundedness of a semigroup in terms of multiplier properties of the resolvent. Theorem 1.1 is then deduced using Plancherel's theorem, known connections between Fourier multipliers and analytic semigroups from Batty and Srivastava (2003), and a Fourier multiplier theorem for positive kernels from Proposition 3.7.

## 2. Notation and Preliminaries

Apply Theorem 1.1 to obtain optimality of the growth rate in a perturbed wave equation which was studied Denote by $\mathbb{C}_+ := \{\lambda_j \in \mathbb{C} | Re(\lambda_j) > 0\}$ and $\mathbb{C}_- := -\mathbb{C}_+$ the open complexright and left half-planes.

Nonzero Banach spaces over the complex numbers are denoted by $X$ and $Y$. The space of bounded linear sequence operators from $X$ to $Y$ is $\mathcal{L}(X,Y)$, and $\mathcal{L}(X) := \mathcal{L}(X,X)$. The identity sequence of operator on $X$ is denoted by $I_{\mathrm{X}}$, and usually write $\lambda_j$ for $\lambda_j I_{\mathrm{X}}$ when $\lambda_j \in \mathbb{C}$. The domain of a closed sequence of operators $A_j$ on $X$ is $D(A_j)$, a Banach space with thenorm

$$\sum_j \|x^j\|_{D\sum_j(A_j)} := \sum_j \|x^j\|_X + \sum_j \|A_j x^j\|_X \left(x^j \in D(A_j)\right).$$

The sequence of spectrums of $A_j$ are $\sigma(A_j)$ and the resolvent sets are $\rho(A_j) = \mathbb{C} \backslash \sigma(A_j)$. Write $R(\lambda_j, A_j) = (\lambda_j - A_j)^{-1}$ for the resolvent sequence of operators of $A_j$ at $\lambda_j \in \rho(A_j)$.

For $0 \le \epsilon \le \infty$ and $\Omega$ a measure space, $L^{(1+\epsilon)}(\Omega; X)$ is the Bochner space of equivalence classes of strongly measurable, $(1+\epsilon)$-integrable, $X$-valued functions on $\Omega$. The Hölder conjugate of $0 \le \epsilon \le \infty$ is $0 \le \acute{\epsilon} \le \infty$ and is defined .

The indicator function of a set $\Omega$ is denoted by $\Omega_1$. Often identify functions on $[0,\infty)$ with their extension to $\mathbb{R}$ which is identically zero on $(-\infty, 0)$.

The class of $X$-valued Schwartz functions on $\mathbb{R}^n$, $n \in \mathbb{N}$, is denoted by $S(\mathbb{R}^n; X)$, and $S(\mathbb{R}^n) := S(\mathbb{R}^n; \mathbb{C})$. The space of continuous linear $f_j : S(\mathbb{R}^n) \to X$, the$X$-valued tempered distributions, is $S'(\mathbb{R}^n; X)$. The Fourier transform of $f_j \in S'(\mathbb{R}^n; X)$ is denoted by $\mathcal{F} f_j$ or $\hat{f}_j$. If $f_j \in L^1(\mathbb{R}^n; X)$ then

$$\mathcal{F} \sum_j f_j(\xi_j) = \sum_j \hat{f}_j(\xi_j) = \int_{\mathbb{R}^n} \sum_j e^{-i\xi_j \cdot (1+\epsilon)} f_j(1+\epsilon) d(1+\epsilon) \quad (\xi_j \in \mathbb{R}^n).$$

Let $X$ and $Y$ be Banach spaces. A function $m: \mathbb{R}^n \to \mathcal{L}(X,Y)$ is $X$-strongly measurable if $\xi_j \mapsto m(\xi_j)x$ are strongly measurable $Y$-valued map for all $x^j \in X$. Say that m is of moderate growth if there exist $0 < \epsilon < \infty$ and $g^j \in L^1(\mathbb{R})$ such that

$$(1 + |\sum_j \xi_j|)^{-(\beta+\epsilon)} \|m(\xi_j)\|_{\mathcal{L}(X,Y)} \le \sum_j g^j(\xi_j) \qquad (\xi_j \in \mathbb{R}^n).$$

Let $m: \mathbb{R}^n \to \mathcal{L}(X,Y)$ be an $X$-strongly measurable map of moderate growth.

Then $T_m^j : S(\mathbb{R}^n; X) \to S'(\mathbb{R}^n; Y)$,

$$\sum_j T_m^j \ (f_j) := \mathcal{F}^{-1}\left(m \cdot \sum_j \hat{f}_j\right)\left(f_j \in S(\mathbb{R}^n; X)\right), \tag{5}$$

is the Fourier multiplier sequence of operators associated with $m$.

For $0 \le \epsilon < \infty$ and $0 \le \epsilon \le \infty$. Let $\mathcal{M}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R}^n; \mathcal{L}(X,Y))$ be the set of all $X$-strongly

measurable $m: \mathbb{R}^n \to \mathcal{L}(X,Y)$ of moderate growth such that
$T_m^j \in \mathcal{L}\big(L^{(1+\epsilon)}(\mathbb{R}^n; X), L^{(1+\epsilon)}(\mathbb{R}^n;\ Y\ )\big)$, with

$$\|m\|_{\mathcal{M}_{(1+\epsilon),(1+\epsilon)}}(\mathbb{R}^n; \mathcal{L}(X,Y)) := \sum_j \left\|T_m^j\ \right\|_{\mathcal{L}\left(L^{(1+\epsilon)}(\mathbb{R}^n;X), L^{(1+\epsilon)}(\mathbb{R}^n;\ Y\ )\right)}.$$

Furthermore, suppose that there exists an $X$-strongly measurable
$K: \mathbb{R}^n \to \mathcal{L}(X,Y)$ such that $K(\cdot)x^j \in L^1(\mathbb{R}^n; Y)$ and $m(\xi_j)x^j = \mathcal{F}(K(\cdot)x^j)(\xi_j)$ for all $x^j \in X$ and $\xi_j \in \mathbb{R}^n$. Then for $f_j \in L^\infty\ \ (\mathbb{R}^n) \otimes X$ an $X$-valued simple function one may define

$$\sum_j T_m^j\ \ (f_j)(1+\epsilon) := \int_{\mathbb{R}^n} K(0) \sum_j f_j\,(1+\epsilon) d(1+\epsilon)\ (1+\epsilon \in \mathbb{R}^n).$$

Write $m \in \mathcal{M}_{\infty,\infty}(\mathbb{R}^n; \mathcal{L}(Y,X))$ if there exists a constant C ≥ 0 such that

$$\sum_j \left\|T_m^j(f_j)\right\|_{L^\infty(\mathbb{R}^n;Y)} \le C \left\|\sum_j f_j\right\|_{L^\infty(\mathbb{R}^n;X)} \tag{6}$$

for all such $f_j$, and then let $\|m\|_{\mathcal{M}_{\infty,\infty}(\mathbb{R}^n;\mathcal{L}(Y,X))}$ be the minimal constant C in (6). In this case Tm extends to a bounded sequence of operators from the closure of the $X$-valued simple functions in $L^\infty(\mathbb{R}^n; X)$ to $L^\infty(\mathbb{R}^n; Y)$. This closure is not ingeneral equal to $L^\infty(\mathbb{R}^n; X)$, but for $n = 1$ it contains all regulated functions that vanish at infinity, which suffice for the purposes.
For $\varphi_j \in (0,\pi)$ set

$$S_{\varphi_j} := \{z^j \in \mathbb{C}\{0\} \mid |\arg(z^j)| < \varphi_j\}.$$

The sequence of operators $A_j$ on Banach space $X$ is sectorial of angle

$\varphi_j \in (0,\pi)$ if $\sigma(A_j) \subseteq \overline{S_{\varphi_j}}$ and if $sup\left\{\sum_j \left\|\lambda_j R(\lambda_j, A_j)\right\|_{\mathcal{L}(X)} \middle| \lambda_j \in \mathbb{C}\backslash\overline{S_\theta}\right\} < \infty$ for all $\theta \in (\varphi_j, \pi)$.

The sequence of operators $A_j$ such that

$$M\sum_j (A_j) := sup\{\left\|\sum_j \lambda_j\,(\lambda_j + A_j)^{-1}\right\|_{\mathcal{L}(X)} |\lambda_j \in (0,\infty)\} < \infty$$

is sectorial of angle $\varphi_j = \pi - \arcsin(1/M(A_j))$, and for each $\theta > \pi - \arcsin\left(\frac{1}{M(A_j)}\right)$ there exists a constant $C_\theta \ge 0$ independent of $A_j$ such that

$$sup\left\{\left\|\sum_j \lambda_j\, R(\lambda_j, A_j)\right\|_{\mathcal{L}(X)} \middle| \lambda_j \mathbb{C}\backslash\overline{S_\theta}\right\} \le C_\theta M(A_j), \tag{7}$$

as follows from the proof of Haase, (2006), Proposition 2.1.1.a. For $-A_j$ these quence of generators of a $C_0$-semigroup $(T^j(1+\epsilon))_{\epsilon\ge -1} \subseteq \mathcal{L}(X)$ on a Banach space $X$, set

$$\sum_j \omega_0^j(T^j) := \inf\{\omega^j \in \mathbb{R} | \exists M \ge 0: \left\|\sum_j T^j(1+\epsilon)\right\|_{\mathcal{L}(X)} \le M \sum_j e^{\omega^j(1+\epsilon)} \text{ for all } \epsilon \ge -1\}$$

and $(1+\epsilon)(-A_j) := sup\{Re(\lambda_j) | \lambda_j \in \sigma(-A_j)\}$. Then $\omega^j + A_j$ is sectorial sequence of operators for $\omega^j > \omega_0^j\,(T^j)$. In particular, for $\gamma\ \ \in [0,\infty)$ the fractional domain $X := D\left(\left(\omega^j + A_j\right)\right)$ is well defined, and up to series norm equivalence it is independent of the choice of $\omega^j$.For background knowledge on $C_0$-semigroups and sectorial sequence of operators

## 3. Polynomial Growth Results

For $-A_j$ the sequence of generators of a $C_0$-semigroup $(T^j(1+\epsilon))_{\epsilon\geq -1}$ on a Banach space $X$, let $\omega^j, M_{\omega^j}\geq 1$ be such that

$$\left\|\sum_j T^j(1+\epsilon)\right\|_{\mathcal{L}(X)} \leq \sum_j M_{\omega^j} e^{(1+\epsilon)(\omega^j-1)}\ (\epsilon\geq -1), \tag{8}$$

and set $M:=sup\left\{\left\|\sum_j T^j\,(1+\epsilon)\right\|_{\mathcal{L}(X)}\middle|\ 0\leq\epsilon\leq 1\right\}$.

### *3.1 General Banach Spaces*

First consider semigroups on general Banach spaces. In Eisner and Zwart (2007) an example is given of a semigroup sequence of generators $-A_j$ which satisfies (1) such that the associated semigroup grows exponentially. The following proposition shows in specific that the Kreiss condition does imply at most linear growth of semigroup orbits with sufficiently smooth initial values.

**Proposition 3.1**. Let $-A_j$be the sequence of generators of a $C_0$-semigroup $(T^j(1+\epsilon))_{\epsilon\geq -1}$ on a Banachspace $X$ such that $\mathbb{C}_-\subseteq\rho(A_j)$. Assume that there exists a nondecreasing $g^j:(0,\infty)\to(0,\infty)$ such that

$$\left\|\sum_j(\lambda_j+A_j)^{-1}\right\|_{\mathcal{L}(X)}\leq\sum_j g^j\left(Re(\lambda_j)^{-1}\right)\qquad(\lambda_j\in\mathbb{C}_+).$$

Then for each $\gamma\in(1,\infty)$ there exists $(1+\epsilon)_\gamma>0$ such that

$\sum_j\left\|T^j(1+\epsilon)\right\|_{\mathcal{L}(X_\gamma\ \ ,X)}\leq(1+\epsilon)_\gamma\ \ \sum_j g^j\,(1+\epsilon)+M$ for all $\epsilon>-1$.

**Proof**. It suffices to prove the estimate for $\epsilon\geq 1$. Let $x^j\in X$ and set $y^j:=(1+A_j)x^j\in X$. For $\beta+\epsilon\in(0,1)$ the functional calculus for half-plane sequence of operators from Batty, Haase and Mubeen (2013) yields

$$e^{-(1+\epsilon)(1+\epsilon)}\sum_j T^j(1+\epsilon)x^j=\frac{1}{2\pi i}\int_{i\mathbb{R}}\sum_j\frac{e^{-z^j(1+\epsilon)}}{(-\epsilon+z^j)^\gamma}R\left(z^j,A_j+(1+\epsilon)\right)y^j dz^j.$$

Hence there exists a constant $C_\gamma^{'}>0$ such that, for all $(1+\epsilon)\in(0,\frac{1}{2})$,

$$\sum_j\left\|T^j(1+\epsilon)x^j\right\|_X\leq\frac{1}{2\pi}e^{(\epsilon^2+\epsilon+1)}\sum_j g^j\,(1/(1+\epsilon))\left\|y^j\right\|_X\int_{i\mathbb{R}}\frac{1}{|-\epsilon+z^j|^{\gamma_j}}\left|dz^j\right|$$

$$\leq C_\gamma^{'}e^{(\epsilon^2+\epsilon+1)}\sum_j g^j\,(1/(1+\epsilon))\left\|x^j\right\|_X,$$

Set $\epsilon^2+2\epsilon$ to conclude the proof.

**Theorem 3.2**. Let $-A_j$ be the sequence of generators of $C_0$-semigroup $(T^j(1+\epsilon))_{\epsilon\geq -1}$ on a Banachspace $X$ such that $\mathbb{C}_-\subseteq\rho(A_j)$, and let $Y\hookrightarrow X$ be a continuously embedded Banachspace satisfying the following conditions:

(1) There exists $(1+\epsilon)_{T^j} \geq 0$ such that $T^j(1+\epsilon) \in \mathcal{L}(Y)$ for all $\epsilon \geq -1$,

(2) With $\sum_j \left\|T^j(1+\epsilon)\right\|_{\mathcal{L}(Y)} \leq (1+\epsilon)_{T^j} T^j \left\|\sum_j T^j(1+\epsilon)\right\|_{\mathcal{L}(X)}$;

(2) There exists a continuously and densely embedded Banach space $Y_0 \hookrightarrow Y$such that $\left[(1+\epsilon) \mapsto e^{-(1+\epsilon)(1+\epsilon)} \sum_j \left\|T^j(1+\epsilon)\right\|_{\mathcal{L}(Y_0,X)}\right] \in L^1(0,\infty)$ for all $1 < \epsilon < \infty. Assume$ that there exist $0 \leq \epsilon \leq \infty$ and a nondecreasing $g^j: (0,\infty) \to (0,\infty)$ such that $(a + i \cdot + A_j)^{-1} \in M_{(1+\epsilon),(1+\epsilon)}(\mathbb{R}; \mathcal{L}(Y,X))$ for all $1 < \epsilon < \infty$, with

$$\left\|\sum_j ((1+\epsilon) + i \cdot + A_j)^{-1}\right\|_{M_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\mathcal{L}(Y,X))} \leq \sum_j g^j\,(1/(1+\epsilon)). \tag{9}$$

Then $\left\|T^j(1+\epsilon)\right\|_{\mathcal{L}(Y)} \leq C_{(1+\epsilon)}(g^j(1+\epsilon)+1)$ for all $\epsilon > -1$. Here $C_{1+\epsilon} = e(1+\epsilon)_{T^j} C_Y M_{\omega^j}(1+2M\omega^j)$ for $1+\epsilon < \infty, (1+\epsilon)_\infty = e(1+\epsilon)_{T^j} C_Y M_{\omega^j}(1+\omega^j)$, and $C_Y = max(1, \|I_Y\|_{\mathcal{L}(Y,X)})$.

**Proof**. Set $m_{(1+\epsilon)}(\xi_j) := ((1+\epsilon) + i\xi_j + A_j)^{-1} \in \mathcal{L}(Y,X)$ for $\epsilon > -1$ and $\xi_j \in \mathbb{R}$. First prove

$$\left\|m_{(1+\epsilon)}\right\|_{M_{(1+\epsilon),\infty}}\big(\mathbb{R}; \mathcal{L}(Y,X)\big) \leq 2M(g^j(1/(1+\epsilon)) + C_Y) \tag{10}$$

for $\epsilon < \infty$. Let $f_j \in S(\mathbb{R}) \otimes Y_0$ be such that $\left\|\sum_j f_j\right\|_{L^{(1+\epsilon)}(\mathbb{R};Y)} \leq 1$. Then $\left\|\sum_j T^j_{m_{(1+\epsilon)}}(f_j)\right\|_{L^{(1+\epsilon)}(\mathbb{R};X)} \leq \sum_j g^j\,(1/(1+\epsilon))$, so for each $l \in \mathbb{Z}$ there exists a $1+\epsilon \in [l, l+1]$ such that

$$\left\|\sum_j T^j_{m_{(1+\epsilon)}}(f_j)(1+\epsilon)\right\|_X \leq 2\sum_j g^j\,(1/(1+\epsilon)). \tag{11}$$

Fix an $l \in \mathbb{Z}$ and let $1+\epsilon \in [l, l+1]$ be such that (11) holds. Let $0 \leq \epsilon \leq 1$ and note that.

$$\sum_j e^{-i\xi_j(1+\epsilon)}\, e^{-(\epsilon^2+2\epsilon+\epsilon)} T^j(1+\epsilon) \left((1+\epsilon) + i\xi_j + A_j\right)^{-1} x^j$$

$$= \sum_j \left((1+\epsilon) + i\xi_j + A_j\right)^{-1} x^j \int_0^\tau e^{-\left((1+\epsilon)+i\xi_j\right)} T^j(1+\epsilon) x^j\, d(1+\epsilon)$$

for all $\xi_j \in \mathbb{R}$ and $x^j \in X$. Hence

$$e^{-(\epsilon^2+2\epsilon+\epsilon)} \sum_j T^j(\tau)\, T^j_{m_{(1+\epsilon)}}\left(f_j\right)(1+\epsilon)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \sum_j e^{i\xi_j 2(1+\epsilon)}\, e^{-i\xi_j(1+\epsilon)} e^{-(\epsilon^2+2\epsilon+\epsilon)} T^j((1+\epsilon))((1+\epsilon) + i\xi_j + A_j)^{-1} \hat{f}_j\left(\xi_j\right) d\xi_j$$

$$= \sum_j T^j_{m_{(1+\epsilon)}}\left(f_j\right)(2(1+\epsilon)) - \int_0^\tau e^{-\left(\epsilon^2+2\epsilon+1\right)} \sum_j T^j\,(1+\epsilon) f_j(1+\epsilon)(1+\epsilon)\, d(1+\epsilon).$$

Rearranging terms and using (11) and Hölder's inequality, yield

$$\left\|\sum_j T^j_{m_{(1+\epsilon)}}\left(f_j\right)(2(1+\epsilon))\right\|_X \leq 2M\sum_j g^j\left(1/(1+\epsilon)\right)+(1+\epsilon)^{(1+\epsilon)}MC_Y$$

$$\leq 2M\sum_j g^j\left(1/(1+\epsilon)\right)+C_Y.$$

Because $0 \leq \epsilon \leq 1$ and $l \in \mathbb{Z}$ are arbitrary and since $Y_0 \subseteq Y$is dense, (10) follows. This in turn obtains

$$\sum_j \left\|T^j_{I_{Y+\omega^j m_{(1+\epsilon)}}}(f_j)\right\|_{L^\infty(\mathbb{R};X)} \leq C_Y\left\|\sum_j f_j\right\|_{L^\infty(\mathbb{R};Y)} + 2M\sum_j \omega^j\left(g^j(1/(1+\epsilon))+C_Y\right)\left\|f_j\right\|_{L^{(1+\epsilon)}(\mathbb{R};Y)} \tag{12}$$

for $f_j \in L^\infty(\mathbb{R};Y_0)\cap L^{(1+\epsilon)}(\mathbb{R};Y_0)$. Second hand, for $1+\epsilon=\infty$ one has

$$\sum_j \left\|T^j_{I_{Y+\omega^j m_{(1+\epsilon)}}}(f_j)\right\|_{L^\infty(\mathbb{R};X)} \leq C_Y\sum_j\left\|f_j\right\|_{L^\infty(\mathbb{R};Y)}+\omega^j g^j(1/(1+\epsilon))\left\|f_j\right\|_{L^{(1+\epsilon)}(\mathbb{R};Y)} \tag{13}$$

for all piecewise continuous $f \in L^{(1+\epsilon)}(\mathbb{R};Y_0)\cap L^\infty(\mathbb{R};Y_0)$ that vanish at infinity.

Let $x^j \in Y_0$ and set $f_j(1+\epsilon) := e^{-(\epsilon^2+2\epsilon+\epsilon)}T^j(1+\epsilon)x^j$ for $\epsilon \geq -1$. It follows from $\mathbb{C}_- \subseteq \rho(A_j)$and $[1+\epsilon \mapsto e^{-(\epsilon^2+2\epsilon+\epsilon)}T^j(1+\epsilon)x^j] \in L^1([0,\infty);X)$ that

$$\mathcal{F}\left(\left[1+\epsilon \mapsto e^{-(\epsilon^2+2\epsilon+\epsilon)}T^j(1+\epsilon)x^j\right]\right)(\cdot)=\left((1+\epsilon)+i\cdot+A_j\right)^{-1}x^j \text{ and } F\left(f_j\right)(\cdot)=\left((1+\epsilon)+\omega^j+i\cdot+A_j\right)^{-1}x^j. \tag{14}$$

For $\epsilon > -1$ one has, by the assumptions on $Y$,

$$\sum_j\left\|f_j(1+\epsilon)\right\|_Y \leq C_{T^j}\left\|\sum_j e^{-\left(\omega^j+(1+\epsilon)\right)(1+\epsilon)}\,T^j(1+\epsilon)\right\|_{\mathcal{L}(X)}\left\|x^j\right\|_Y \leq \sum_j C_{T^j}M_{\omega^j}e^{-(1+\epsilon)}\left\|x^j\right\|_Y.$$

Therefore $f_j$ is piecewise continuous, vanishes at infinity, and satisfies $\left\|f_j\right\|_{L^{(1+\epsilon)}(\mathbb{R}_+;Y)} \leq C_{T^j}M_{\omega^j}\left\|x^j\right\|_Y$ for $1+\epsilon \in \{1+\epsilon,\infty\}$. Also, by (14) and the resolvent identity,

$$e^{-(\epsilon^2+2\epsilon+1)}\sum_j T^j(1+\epsilon)x^j=\sum_j T^j_{I_{Y+\omega^j m_{(1+\epsilon)}}}(f_j)(1+\epsilon).$$

Then (12) yields

$$e^{-(\epsilon^2+2\epsilon+1)}\left\|\sum_j T^j(1+\epsilon)x^j\right\|_X \leq \sum_j (1+\epsilon)_{T^j}C_Y M_\omega(1+2M\omega^j)(g^j(1/(1+\epsilon))+1)\left\|x^j\right\|_Y,$$

and (13) implies

$$e^{-(\epsilon^2+2\epsilon+1)}\left\|\sum_j T^j\,(1+\epsilon)x^j\right\|_X \leq \sum_j (1+\epsilon)_{T^j}C_Y M_{\omega^j}(1+\omega^j)(g^j(1/(1+\epsilon))+1)\left\|x^j\right\|_Y,$$

Since $Y_0 \subseteq Y$ is dense, the proof is concluded by setting $\epsilon^2+2\epsilon$.

**Remark 3.3**. Note from the proof of Theorem 3.2 that if there exist $(1+\epsilon)_0 \in (0,\infty)$, $0 \le \epsilon \le \infty$, and a nondecreasing $g^j: (0,\infty) \to (0,\infty)$ such that (9) holds for all $(1+\epsilon) \in (0,(1+\epsilon)_0)$, then $\left\|\sum_j T^j\,(1+\epsilon)\right\|_{\mathcal{L}(Y,X)} \le C\left(\sum_j g^j\,(1+\epsilon)+1\right)$ for all $1+\epsilon > 1/(1+\epsilon)_0$. This will be used inthe proof of Theorem 3.6.

*3.2 Hilbert Spaces*

Apply Theorem 3.2 by bounding the $\mathcal{M}_{(1+\epsilon),(1+\epsilon)}$ norm in (9) by a supremum norm of $((1+\epsilon)+i\cdot +A_j)^{-1}$. First peeking the Hilbert space setting, where the following theorem, in the special case where $g^j$ are polynomial. More general $g^j$ were regarded, where abound of the form $\left\|\sum_j T^j\,(1+\epsilon)\right\|_{\mathcal{L}(X)} \le \frac{\sum_j C g^j(1+\epsilon)^2}{(1+\epsilon)}$ was obtained. Note that $g^j$ which grow sublinearly lead to exponentially stable semigroups.

**Theorem 3.4**. Let $-A_j$ be the sequence of generators of $C_0$-semigroup $(T^j(1+\epsilon))_{\epsilon\ge -1}$ on a Hilbert space $X$ such that $\mathbb{C}_- \subseteq \rho(A_j)$. Suppose that there exists a non decreasing $g^j$:$(0,\infty) \to (0,\infty)$ such that

$$\left\|\sum_j(\lambda_j+\ A_j)^{-1}\right\|_{\mathcal{L}(X)} \le \sum_j g^j\left(Re\left(\lambda_j\right)^{-1}\right)\left(\lambda_j \in \mathbb{C}_+\right). \tag{15}$$

Then $\left\|\sum_j T^j\,(1+\epsilon)\right\|_{\mathcal{L}(X)} \le e\sum_j M_{\omega^j}\,(1+2M\omega^j)(g^j(1+\epsilon)+1)$ for all $\epsilon > -1$.

**Proof**. Condition (2) in Theorem 3.2, with $Y_0 = X_2$ and $Y = X$, is satisfied by Proposition 3.1. Furthermore, Plancherel's identity yields

$$\left\|\sum_j\left((1+\epsilon)+\ i\ \cdot\ +A_j\right)^{-1}\right\|_{\mathcal{M}_{2,2}(\mathbb{R};\mathcal{L}(X))} = \left\|\sum_j((1+\epsilon)+\ i\ \cdot\ +A_j)^{-1}\right\|_{L^\infty(\mathbb{R};\mathcal{L}(X))}$$

$$\le \sum_j g^j\,(1/(1+\epsilon)),$$

so that Theorem 3.2 concludes the proof.

The following example shows that for $g^j$apolynomial, Theorem 3.4 is optimal up to arbitrarily small polynomial loss.

**Example 3.5**. Fix $\gamma\ \in (0,1)$ and $n \in \mathbb{N}$. It is shown that there exist a Hilbert space $X$, $C_0$-semigroup $(S(1+\epsilon))_{\epsilon\ge -1} \subseteq \mathcal{L}(X)$ with boundedsequence of generators $-A_j$, and constants $C_1, C_2 \ge 0$ such that $\sigma(A_j) \subseteq \overline{\mathbb{C}_+}$,

$$\left\|R\sum_j(\lambda_j, A_j)\right\|_{\mathcal{L}(X)} \le \frac{C_1}{Re\sum_j(\lambda_j)} \qquad (\lambda_j \in \mathbb{C}_-)$$

And $\|S(1+\epsilon)\|_{\mathcal{L}(X)} \ge C_2((1+\epsilon)^\gamma\ \ +1)$ for all $\epsilon \ge -1$. Let $J \in \mathcal{L}(X^n)$ be the $n \times n$ sequence of operators matrix with $J_{k,k+1} = -I_X$ for $k \in \{1,\ldots,n-\}$, and $J_{k,l} = 0$ for $l\ \neq k+1$. Set $A_j := A_j(I_{X^n}+J)$, and let $\left(T^j(1+\epsilon)\right)_{\epsilon\ge -1} \subseteq \mathcal{L}(X^n)$ be the $C_0$-semigroup sequence generated by $-A_j$.

Then $T^j(1+\epsilon) = S(1+\epsilon)e^{-(1+\epsilon)}$ for $\epsilon \geq -1$, and $\left\|\sum_j T^j(1+\epsilon)\right\|_{\mathcal{L}(X^n)} \geq c((1+\epsilon)^{\gamma \ +n-1}+1)$ for some $c > 0$ independent of $(1+\epsilon)$. Furthermore, there exists a $\epsilon \geq -1$ such that $\left\|\sum_j(\lambda_j + A_j)^{-1}\right\|_{\mathcal{L}(X^n)} \leq (1+\epsilon)\sum_j(Re(\lambda_j)^{-n}+1)$ for all $\lambda_j \in \mathbb{C}_+$.

*3.3 Asymptotically Analytic Semigroups*

For $C_0$-semigroup $(T^j(1+\epsilon))_{\epsilon\geq -1}$ with sequence of generators $-A_j$ on Banach space $X$, the non-analytic growth series bounds are

$$\zeta\sum_j(T^j) := in\left\{\sum_j \omega^j \in \mathbb{R}\left|\sup_{\epsilon>-1} e^{-\omega^j(1+\epsilon)}\left\|T^j(1+\epsilon) - S(1+\epsilon)\right\| < \infty \text{ for some } \atop S \in \mathcal{H}(\mathcal{B}_j(X))\right.\right\},$$

Where $\mathcal{H}(\mathcal{B}_j(X)$ is the set of $S: (0,\infty) \to \mathcal{B}_j(X)$ having an exponentially bounded analytic extension to some sector containing $(0,\infty)$. Let $(1+\epsilon)_0^\infty(-A_j)$ be the infimumover all $\omega^j \in \mathbb{R}$ for which there exists an $\mathbb{R} \in (0,\infty)$ such that
$\{\eta + i\xi_j | \eta > \omega^j, \xi_j \in \mathbb{R}, |\xi_j| \geq R\} \subseteq \rho(-A_j)$ and $sup\{\left\|(\eta + i\xi_j + A)^{-1}\right\|_{\mathcal{L}(X)} | \eta > \omega^j, \xi_j \in \mathbb{R}, |\xi_j| \geq R\} < \infty$.
If $\zeta(T^j) < 0$ then $(T^j(1+\epsilon))_{\epsilon\geq -1}$ are asymptotically analytic. Then $(1+\epsilon)_0^\infty(-A_j)$, and theconverse implication holds if X is a Hilbert space. It is trivial that if $(T^j(1+\epsilon))_{\epsilon\geq -1}$ is analytic semigroup then $\zeta(T^j) = -\infty$. In fact, $\zeta(T^j) = -\infty$ if $(T^j(1+\epsilon))_{\epsilon\geq -1}$ is eventually differentiable. For more on asymptotically analytic semigroups see Batty and Srivastava (2003).

**Theorem 3.6**. Let $-A_j$ be the sequence of generators of an asymptotically analytic $C_0$-semigroup $(T^j(1+\epsilon))_{\epsilon\geq -1}$ on a Banach space $X$ such that $\mathbb{C}_- \subseteq \rho(A_j)$. Suppose that there exists anon decreasing $g^j: (0,\infty) \to (0,\infty)$ such that

$$\left\|\sum_j(\lambda_j + A_j)^{-1}\right\|_{\mathcal{L}(X)} \leq \sum_j g^j\left(Re(\lambda_j)^{-1}\right) \qquad (\lambda_j \in \mathbb{C}_+).$$

Then there exists $\epsilon \geq -1$ such that $\left\|\sum_j T^j(1+\epsilon)\right\|_{\mathcal{L}(X)} \leq (1+\epsilon)(\sum_j g^j(1+\epsilon)+1)$ for all $\epsilon > -1$.

**Proof**. There exist $(1+\epsilon)_0 > 0$ and $\psi_j \in C_c^\infty(\mathbb{R})$ such that
$(1-\psi_j(\cdot))((1+\epsilon)+i\cdot+A_j)^{-1} \in \mathcal{M}_{1,\infty}(\mathbb{R};\mathcal{L}(X))$ for all $(1+\epsilon) \in (0,(1+\epsilon)_0)$, with

$$C_1 := sup\{\left\|(1-\psi_j(\cdot))((1+\epsilon)+i\cdot+A_j)^{-1}\right\|_{\mathcal{M}_{1,\infty}}(\mathbb{R};\mathcal{L}(X)) | (1+\epsilon) \in (0,(1+\epsilon)_0)\} < \infty.$$

Second hand, a straightforward series estimates shows that
$\psi_j(\cdot)((1+\epsilon)+i\cdot+A_j)^{-1} \in \mathcal{M}_{1,\infty}(\mathbb{R};\mathcal{L}(X))$ for all $\epsilon > -1$, with

$$\sum_j\left\|\psi_j(\cdot)((1+\epsilon)+i\cdot+A_j)^{-1}\right\|_{\mathcal{M}_{1,\infty}(\mathbb{R};\mathcal{L}(X))} \leq \frac{1}{2\pi}\left\|\sum_j \psi_j(\cdot)\,((1+\epsilon)+i\cdot+A_j)^{-1}\right\|_{L^1(\mathbb{R};\mathcal{L}(X))}$$

$$\leq (1+\epsilon)_2\sum_j g^j\,(1/(1+\epsilon))$$

for some $(1+\epsilon)_2 \geq 0$ independent of $(1+\epsilon)$. It follows that

$$\left\|\sum_j ((1+\epsilon)+i\cdot+A_j)^{-1}\right\|_{\mathcal{M}_{1,\infty}(\mathbb{R};\mathcal{L}(X))} \leq C_1+\frac{1}{2\pi}\sum_j g^j\,(1/(1+\epsilon))$$

$$\leq C_3 g^j(1/(1+\epsilon))\big((1+\epsilon)\in(0,(1+\epsilon)_0)\big),$$

where $C_3 = C_1 g^j(1/(1+\epsilon)_0)^{-1}+(1+\epsilon)_2$. Then Remark 3.3 yields a constant $C'\geq 0$ suchthat $\left\|\sum_j T^j\,(1+\epsilon)\right\|_{\mathcal{L}(X)} \leq C'(\sum_j g^j\,(1+\epsilon)+1)$ for all $1+\epsilon>1/(1+\epsilon)_0$. Since $sup\{\sum_j\left\|T^j(1+\epsilon)\right\|_{\mathcal{L}(X)} | 1+\epsilon\in[0,1/(1+\epsilon)_0]\}<\infty$, this concludes the proof.

*3.4 Positive Semigroups*

Peek positive $C_0$-semigroups on various Banach lattices. To this end first prove a multiplier theorem for positive kernels. Part of this result is already contained. Recall that a subspace $X$ of a Banach lattice $Y$ is a sublatticeif $x^j\vee y^j, x^j\wedge y^j\in X$ for all $x^j,y^j\in X$.

**Proposition 3.7**. Let $n\in\mathbb{N}$, $0\leq\epsilon\leq\infty$, and let $X$ be a Banach lattice and $m:\mathbb{R}^n\to\mathcal{L}(X)$ an $X$-strongly measurable map of moderate growth. Let $K:\mathbb{R}^n\to\mathcal{L}(X)$ be such that $K(\cdot)x^j\in L^1(\mathbb{R}^n;X)$ and $m(\xi_j)x^j=\mathcal{F}(K(\cdot)x^j)(\xi_j)$ for all $x^j\in X$ and $\xi_j\in\mathbb{R}^n$, and such that $K(1+\epsilon)$ is a positive sequence of operators for all $(1+\epsilon)\in\mathbb{R}^n$. Suppose that one of the following conditions holds:

(1) $X=L^{(1+\epsilon)}(\Omega)$ for a measure space;

(2) $1+\epsilon=\infty$ and $X$ is a closed subspace of $C_{(1+\epsilon)}(\Omega)$, for a topological space, such that either $1_\Omega\in X$or $X$ is a sublattice. Then $m\in\mathcal{M}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R}^n;\mathcal{L}(X))$ with

$$\|m\|_{\mathcal{M}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R}^n;\mathcal{L}(X)}=\|m(0)\|_{\mathcal{L}(X)}.$$

**Proof**. It is well known that

$$\|m\|_{\mathcal{M}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R}^n;\mathcal{L}(X)}\geq\sup_{\xi_j\in\mathbb{R}^n}\left\|m\sum_j(\xi_j)\right\|_{\mathcal{L}(X)}\geq\|m(0)\|_{\mathcal{L}(X)}$$

if $m\in\mathcal{M}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R}^n;\mathcal{L}(X))$. In the case where $X=L^{(1+\epsilon)}(\Omega)$ for $0\leq\epsilon<\infty$ it follows that $m\in\mathcal{M}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R}^n;\mathcal{L}(X))$ with the required series estimate.

Next, suppose that $\epsilon=\infty$ and let $f_j:=\sum_k^m \mathbf{1}_{E_k}\otimes(x^j)_k$ for $m\in\mathbb{N}$, $E_1,\dots,E_n\subseteq\mathbb{R}^n$ disjoint and measurable, and $(x^j)_1,\dots,(x^j)_n\in X$. If $\mathbf{1}_\Omega\in X$ set $\sum_j g^j\equiv\sum_j\left\|f_j\right\|_{L^\infty(\mathbb{R}^n;X)}$, and for $X$ a sublattice set $g^j=\vee_{1\leq k\leq m}|(x^j)_k|$. In both cases $\sum_j g^j\in X, \left|\sum_j f_j\,(1+\epsilon)\right|\leq\sum_j g^j$ for all $(1+\epsilon)\in\mathbb{R}^n$, and $\sum_j\left\|f_j\right\|_{L^\infty(\mathbb{R}^n;X)}=\sum_j\left\|g^j\right\|_X$. Then

$$\left|\sum_j T_m^j\,(1+\epsilon)(f_j)\right|\leq\int_{\mathbb{R}^n}\left|K(1+\epsilon)\sum_j f_j(0)\right|d(1+\epsilon)\leq\int_{\mathbb{R}^n}K(1+\epsilon)\sum_j g^j\,d(1+\epsilon)=m(0)g^j$$

for all $(1+\epsilon)\in\mathbb{R}^n$. Hence

$$\sum_j\left\|T_m^j(f_j)\right\|_{L^\infty(\mathbb{R}^n;X)}\leq\|m(0)\|_{\mathcal{L}(X)}\left\|\sum_j g^j\right\|_X=$$

$$\|m(0)\|_{\mathcal{L}(X)} \sum_j \|f_j\|_{L^\infty(\mathbb{R}^n;X)},$$

which concludes the proof.

**Theorem 3.8**. Let $-A_j$ be the sequence of generators of a positive $C_0$-semigroup $(T^j(1+\epsilon))_{\epsilon\geq -1}$ on a Banach lattice $X$ such that $\mathbb{C}_- \subseteq \rho(A_j)$. Assume that one of the following conditions holds:

(1) $X = L^{(1+\epsilon)}(\Omega)$ for $0 \leq \epsilon \leq \infty$ and $\Omega$ a measure space;

(2) $1+\varepsilon = \infty$ and $X$ is a closed subspace of $C_{(1+\epsilon)}(\Omega)$, for a topological space, such that either $\mathbf{1}_\Omega \in X$or $X$ is a sublattice.

Suppose that there exists a non decreasing $g^j: (0,\infty) \to (0,\infty)$ such that

$$\left\|\sum_j ((1+\epsilon)+A_j)^{-1}\right\|_{\mathcal{L}(X)} \leq \sum_j g^j\,(1/(1+\epsilon))\big((1+\epsilon)\in(0,\infty)\big). \tag{16}$$

Then $\left\|\sum_j T^j\,(1+\epsilon)\right\|_{\mathcal{L}(X)} \leq C\sum_j (g^j(1+\epsilon)+1)$for all $\epsilon > -1$, where $C = eM_{\omega^j}(1+2M\omega^j)$ for (1), and $C = eM_{\omega^j}(1+\omega^j)$ if (2) holds.

**Proof**. Set $1+\varepsilon = \infty$ if (2) holds. Let $\epsilon > -1$. First claim that $[1+\epsilon \mapsto e^{-(\epsilon^2+2\epsilon+1)}T^j(1+\epsilon)x^j] \in L^1([0,\infty);X)$ for all $x^j \in X$, with

$$\mathcal{F}\big(\big[1+\epsilon \mapsto e^{-(\epsilon^2+2\epsilon+1)}T^j(1+\epsilon`)x^j\big]\big)(\xi_j) = \Big((1+\epsilon)+i\xi_j+A_j\Big)^{-1} x^j \ \ (\xi_j \in \mathbb{R}).$$

To prove this let $n \geq 2\omega^j$and $(1+\epsilon) \in (0, min((1+\epsilon),\omega^j))$, and set $\left(B_j\right)_n := n^2(n + A_j)^{-2}$ and $K_{n,(1+\epsilon)}(1+\epsilon) := e^{-(\epsilon^2+2\epsilon+1)}T^j(1+\epsilon)\left(B_j\right)_n$ for $\epsilon \geq -1$. Then $K_{n,(1+\epsilon)}(1+\epsilon)$ is a positive sequence of operator for all $\epsilon \geq -1$, and $K_{n,(1+\epsilon)}(\cdot)x^j \in L^1(\mathbb{R};X)$ with

$$\mathcal{F}\big(K_{n,(1+\epsilon)}(\cdot)x^j\big)\sum_j(\xi_j) = \sum_j \Big((1+\epsilon)+i\xi_j+A_j\Big)^{-1}\left(B_j\right)_n x^j(\xi_j \in \mathbb{R}),$$

where use Proposition 3.1. By Proposition 3.7, $((1+\epsilon)+i\cdot+A)^{-1}\left(B_j\right)_n \in \mathcal{M}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\mathcal{L}(X))$ with

$$\left\|\sum_j((1+\epsilon)+i\cdot+A_j)^{-1}\left(B_j\right)_n\right\|_{\mathcal{M}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\mathcal{L}(X))} \leq 4\sum_j g^j\,(1/(1+\epsilon))M_{\omega^j}^2, \tag{17}$$

where used (8) to deduce that $\left\|n\sum_j(n + A_j)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{n}{n-\omega+1}\sum_j M_{\omega^j} \leq 2\sum_j M_{\omega^j}$.

Let $x^j \in X$ and set $f_j(1+\epsilon) := e^{-\omega^j(1+\epsilon)}T^j(1+\epsilon)x^j$ for $\epsilon \geq -1$. Then $f_j \in L^{(1+\epsilon)}(\mathbb{R};X) \cap L^1(\mathbb{R};X)$ is piecewise continuous and vanishes at infinity, and

$K_{n,(1+\epsilon)} * \sum_j f_j = \sum_j T^j_{((1+\epsilon)+i\cdot+A_j)^{-1}(B_j)_n}\,(f_j)$.

Moreover,

$$\begin{aligned} K_{n,(1+\epsilon)} * \sum_j f_j(1+\epsilon) &= \int_0^{(1+\epsilon)} \sum_j e^{-\left(\omega^j-(1+\epsilon)\right)(1+\epsilon)}\, e^{-(\epsilon^2+2\epsilon+1)}T^j(1+\epsilon)(B_j)_n\ x^j d(1+\epsilon) \\ &= \sum_j \frac{1-e^{-\left(\omega^j-(1+\epsilon)\right)(1+\epsilon)}}{\omega^j-(1+\epsilon)}\, e^{-(\epsilon^2+2\epsilon+1)}\,T^j(1+\epsilon)(B_j)_n\ x^j. \end{aligned}$$

Since $\left(B_j\right)_n \to I_X$ strongly as $n \to \infty$, (17) yields a constant $C_{(1+\epsilon)} \geq 0$ such that

$$e^{-(\epsilon^2+2\epsilon+1)}\left\|\sum_j T^j\,(1+\epsilon)x^j\right\|_X \le C_{(1+\epsilon)}\sum_j\left\|x^j\right\|_X \qquad (\epsilon \ge 0).$$

This shows that $[1+\epsilon \mapsto e^{-(\epsilon^2+2\epsilon+1)}T^j(1+\epsilon)x^j] \in L^1([0,\infty);X)$ for all $x^j \in X$, and the identity

$$\mathcal{F}\sum_j\left(\left[1+\epsilon \mapsto e^{-(\epsilon^2+2\epsilon+1)}T^j(1+\epsilon)x^j\right]\right)(\xi_j) =$$

$$\sum_j\left((1+\epsilon)+i\xi_j+A_j\right)^{-1}x^j \qquad (\xi_j \in \mathbb{R})$$

is then straightforward. This proves the claim.

Finally, since $e^{-(\epsilon^2+2\epsilon+1)}T^j(1+\epsilon)$ is a positive sequence of operators for all $\epsilon \ge -1$, Proposition 3.7 yields $((1+\epsilon)+i\cdot+A_j)^{-1} \in \mathcal{M}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\mathcal{L}(X))$ with

$$\sum_j\left\|((1+\epsilon)+i\cdot+A_j)^{-1}\right\|_{\mathcal{M}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\mathcal{L}(X))} = \left\|\sum_j((1+\epsilon)+A_j)^{-1}\right\|_{\mathcal{L}(X)} \le \sum_j g^j\,(1/(1+\epsilon)).$$

Then Theorem 3.2 concludes the proof.

Theorem 3.8 implies in particular that $\omega_0^j\,(T^j) = (1+\epsilon)(-A_j)$ for a positive semigroup $(T^j(1+\epsilon))_{\epsilon\ge-1}$ on a space $X$as in (1) or (2). For (1) this result was originally obtained. It is possible to extend Theorem 3.8 to fractional domains on more general Banach lattices, by using Fourier multipliers on $X$-valued Besov spaces, but not pursue this matter here.

Do not know whether the growth rate in Theorem 3.8 is optimal. It follows that the positivity assumption cannot be dropped in case (1) for $\epsilon \ne 1$. Furthermore, shows that Theorem 3.8 is not valid on $X = L^{(1+\epsilon)}(\Omega) \cap L^{(1+\epsilon)}(\Omega)$ for $\Omega$ a measure space and $0 \le \epsilon < \infty$,

*3.5 Fourier and Rademacher Type*

Enhance Proposition 3.1 under additional geometric assumptions on $X$. A Banach space $X$ is said to have Fouriertype $0 \le \epsilon \le 1$ if the Fourier transform $\mathcal{F}$ is bounded from $L^{(1+\epsilon)}(\mathbb{R};X)$ into $L^{\left(\frac{1+\epsilon}{\epsilon}\right)}(\mathbb{R};X)$. See Haase (2006) for more on Fourier type. Note in particular that $L^{u^j}(\Omega)$, for $\Omega$ a measure space and $u^j \in [1,\infty]$, has Fourier type $(1+\epsilon) = min(u^j, \acute{u}^j)$.

**Proposition 3.9**. Let $-A_j$be the sequence of generators of $C_0$-semigroup $(T^j(1+\epsilon))_{\epsilon\ge-1}$ on a Banachspace $X$with Fourier type $0 \le \epsilon \le 1$ such that $\mathbb{C}_- \subseteq \rho(A_j)$. Assume that there exists a non decreasing $g^j\colon(0,\infty) \to (0,\infty)$ such that

$$\left\|\sum_j(\lambda_j+A_j)^{-1}\right\|_{\mathcal{L}(X)} \le \sum_j g^j\,\left(Re(\lambda_j)^{-1}\right) \qquad (\lambda_j \in \mathbb{C}_+).$$

Then for each $\gamma \in (\frac{1-\varepsilon}{1+\varepsilon},\infty)$ there exists $(1+\epsilon)_\gamma \ge 0$ such that

$\left\|\sum_j T^j\,(1+\epsilon)\right\|_{\mathcal{L}(X_{\gamma_j},X)} \le \sum_j(1+\epsilon)_\gamma\ \ (g^j(1+\epsilon)+1)$ for all $\epsilon > -1$. For $\epsilon = 1$ one may let $\gamma = 0$.

**Proof**. The case where $\epsilon = 0$ follows from Proposition 3.1. Hence assume that $\gamma \in [0,1)$, and also suppose that $\sum_j g^j(1+\epsilon) > c$ for all $\epsilon > -1$ and some $c > 0$. Then (8) yields

$$\sup_{\lambda_j > 2\omega^j} \lambda_j \left\| \sum_j \left(\lambda_j + A_j + (1+\epsilon)\right)^{-1} \right\|_{\mathcal{L}(X)} \leq 2 \sum_j M_{\omega^j} \leq 2c^{-1} \sum_j M_{\omega^j} g^j\,(1/(1+\epsilon)) \quad (\epsilon > -1).$$

Therefore $A_j + (1+\epsilon)$ is an injective sectorial operator, there exists a $(1+\epsilon)_1 \geq 0$ independent of such that

$$\sup_{\lambda_j \notin \overline{S_\theta}} \left\| \sum_j \lambda_j R(A_j + (1+\epsilon)) \right\|_{\mathcal{L}(X)} \leq (1+\epsilon)_1 \sup_{\lambda_j > 0} \sum_j \left\| \lambda_j (\lambda_j + A_j + (1+\epsilon))^{-1} \right\|_{\mathcal{L}(X)}$$

$$\leq 2(1+\epsilon)_1 \sum_j \left(c^{-1} M_{\omega^j} + \omega^j\right) g^j\,(1/(1+\epsilon)),$$

by (7). It follows from the proof of Batty, Haase and Mubeen (2013), Proposition 3.4 applied to the sequence of operators $A_j + (1+\epsilon)$, by keeping track of the relevant constants, that

$$\left\| \sum_j \left((1+\epsilon) + i\xi_j + A_j\right)^{-1} \right\|_{\mathcal{L}\left(X_{\gamma_j}, X\right)} \leq (1+\epsilon)_2 \sum_j \left(1 + |\xi_j|\right)^{-\gamma_j} g^j(1/(1+\epsilon)) \quad (\xi_j \in \mathbb{R})$$

for some $(1+\epsilon)_2 \geq 0$. So Rozendaal and Veraar (2018), Proposition 3.9 yields constants $C_3, C_4 \geq 0$ such that,

$$\sum_j \left\| (a + i\cdot + A_j)^{-1} \right\|_{\mathcal{M}_{(1+\epsilon),\left(\frac{1+\epsilon}{\epsilon}\right)}(\mathbb{R};\mathcal{L}(X_{\gamma_j},X))} \leq C_3 \left\| \sum_j ((1+\epsilon) + i\cdot + A_j)^{-1} \right\|_{L^{\gamma}(\mathbb{R};\mathcal{L}(X_{\gamma_j},X))}$$

$$\leq C_4 \sum_j g^j\,(1/(1+\epsilon)).$$

Let $Y := X_{\gamma_j}$ and $Y_0 := X_2$ in Theorem 3.2, using Proposition 3.1.

Identical result holds under type and cotype assumptions on the underlying space, and $R$-boundedness assumptions on the resolvent. Let $((1+\epsilon)_k)_{k\in\mathbb{N}}$ be a sequence of independent real Rademacher variables on some probability space. Let $X$and $Y$ be Banach spaces and $\mathcal{T}^j \subseteq \mathcal{L}(X,Y)$. Say that $\mathcal{T}^j$are R -boundedif there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}, T_1^j, \ldots, T_n^j \in \mathcal{T}^j$ and $(x^j)_1, \ldots, (x^j)_n \in X$ one has

$$\left(\mathbb{E}\left\| \sum_{k=1}^n (1+\epsilon)_k T_k^j (x^j)_k \right\|_Y^2\right)^{1/2} \leq C \left(\mathbb{E}\left\| \sum_{k=1}^n (1+\epsilon)_k \overset{x}{(x^j)_k} \right\|_X^2\right)^{1/2}.$$

The smallest such $C$ is the $R$-bound of $\mathcal{T}^j$ and are denoted by $R(\mathcal{T}^j)$. When want to specify the underlying spaces $X$ and $Y$ we write $R_{X,Y}(\mathcal{T}^j)$ for the $R$-bound of $\mathcal{T}^j$, and write $R_X(\mathcal{T}^j) := R_{X,Y}(\mathcal{T}^j)$ if $X = Y$.

For the definitions of and background on type and cotype, and for $(1+\epsilon)$-convexity and $(1+\epsilon)$-concavity of Banach lattices. Note that $X = L^{u^j}(\Omega)$, for $0 \le \epsilon < \infty$ and $\Omega$ a measure space, has type $(1+\epsilon) = min(u^j, 2)$ and cotype$(1+\epsilon)$= max(2, $u^j$) and is $u^j$-convex and $u^j$-concave. For such $X$ the first statement of the following proposition yields the same conclusion as Proposition 3.9.

**Proposition 3.10**. Let $-A_j$ be the sequence of generators of $C_0$-semigroup $(T^j(1+\epsilon))_{\epsilon \ge -1}$ on aBanach space $X$with type $0 \le \epsilon \le \infty$ and cotype $0 \le \epsilon < \infty$ such that $\mathbb{C}_- \subseteq \rho(A_j)$. Suppose that there exists a non decreasing $g^j : (0,\infty) \to (0,\infty)$ such that

$$\left\| \sum_j (\lambda_j + A_j)^{-1} \right\|_{\mathcal{L}(X)} \le \sum_j g^j \left(Re(\lambda_j)^{-1}\right) (\lambda_j \in \mathbb{C}_+).$$

Then for each $\gamma \in (0,\infty)$ there exists $(1+\epsilon)_\gamma \ge 0$ such that $\left\|\sum_j T^j\ (1+\epsilon)\right\|_{\mathcal{L}(X_{\gamma_j}, X)} \le \sum_j (1+\epsilon)_\gamma \ \ (g^j(1+\epsilon)+1)$ for all $\epsilon > -1$. If

$$R_X\left(\left\{\sum_j \left((1+\epsilon) + i\xi_j + A_j\right)^{-1} \middle| \xi_j \in \mathbb{R}\right\}\right) \le \sum_j g^j\,(1/(1+\epsilon)) \qquad ,(1 < \epsilon < \infty)$$

then one may let $\gamma_j \in (0,\infty)$. If in addition $X$ is a $(1+\epsilon)$-convex and $(1+\epsilon)$-concave Banach lattice then one may let $\gamma = 0$.

Let $\epsilon = \infty$ in the first two statements in this proposition. Whatever, then Proposition 3.1 yields a stronger statement, since any Banach space has type $\epsilon = 0$ and cotype $\epsilon = \infty$, and because a Banach space that does not have finitecotype also does not have nontrivial type.

**Proof**. Suppose that $\gamma \in (0,1)$, by Proposition 3.1 and because each 2-convex and 2-concave Banach lattice is isomorphic to a Hilbert space. Also suppose that $g^j(1+\epsilon) > c$ for all $\epsilon > -1$ and some $c > 0$. First prove the final two statements.

As in the proof of Proposition 3.9, it suffices to check the multiplier condition in Theorem 3.2. Furthermore, again using series estimates in proceeding as in the proof of Proposition 3.9, one yield a $(1+\epsilon)_1 \ge 0$ such that

$$R_{X_\gamma ,X}\left(\left\{\left(1+|\xi_j|\right)^\gamma \ \left((1+\epsilon) + i\xi_j + A_j\right)^{-1} \middle| \xi_j \in \mathbb{R}\right\}\right) \le$$

$$(1+\epsilon)_1 g^j(1/(1+\epsilon)) \qquad (\epsilon > -1).$$

yield $(1+\epsilon)_2 \ge 0$ such that

$$\left\| \sum_j \left((1+\epsilon) + i \cdot + A_j\right)^{-1} \right\|_{\mathcal{M}_{(1+\epsilon),(1+\epsilon)}\left(\mathbb{R};\mathcal{L}\left(X_{\gamma_j},X\right)\right)} \le (1+\epsilon)_2 \sum_j g^j\,(1/(1+\epsilon))$$

$$(\epsilon > -1),$$

which proves the final two statements.

For the first statement we may assume that $0 < 1$ and show that for all $\gamma \in (0,1)$ there exists

$(1+\epsilon)_3 \geq 0$ such that

$$R_{X_\gamma \ ,X}\left(\left\{\left(1+\left|\sum_j \xi_j\right|\right)^{\gamma \ /2}\left((1+\epsilon)+i\xi_j+A_j\right)^{-1}\middle|\xi_j\in\mathbb{R}\right\}\right)\leq(1+\epsilon)_3\sum_j g^j\,(1/(1+\epsilon))(\epsilon>-1), \tag{18}$$

after which one proceeds as before. To obtain (18) let $0 \leq \epsilon \leq \infty$ and set $\sum_j (f_j)_{(1+\epsilon)}(\xi_j) := \sum_j (1+|\xi_j|)^{\gamma \ /2}((1+\epsilon)+i\xi_j+A_j)^{-1}$for $\xi_j \in \mathbb{R}$. Then $(f_j)_{(1+\epsilon)} \in W^{1,(1+\epsilon)}(\mathbb{R};\mathcal{L}(X_{\gamma_j},X))$, with

$$\left\|\sum_j (f_j)_{(1+\epsilon)}\right\|_{W^{1,(1+\epsilon)}(\mathbb{R};\mathcal{L}(X_{\gamma_j},X))}\leq(1+\epsilon)_4\sum_J g^j\,(1/(1+\epsilon))$$

for some $(1+\epsilon)_4 \geq 0$ independent of $(1+\epsilon)$, yields (18).

*3.6 Necessary Conditions*

Here provide a converse to Theorem 3.2. For simplicity restrict to semigroups of polynomial growth and to fractional domains, but from the proof one can derive an analogous statement for more general semigroups and more general continuously embedded spaces.

**Theorem 3.11**. Let $-A_j$ be the sequence of generators of $C_0$-semigroup $(T^j(1+\epsilon))_{\epsilon\geq -1}$ on a Banach space $X$. Let $\gamma_j \in [0,\infty)$. Assume that there exist $\epsilon \geq -\beta, \epsilon \geq -1$ such that $\left\|\sum_J T^j(1+\varepsilon)\right\|_{\mathcal{L}(X_{\gamma_j},X)} \leq (1+\epsilon)((1+\epsilon)^{(\beta+\epsilon)}+1)$ for all $\epsilon \geq -1$. Then $\mathbb{C}_- \subseteq \rho(A_j)$ and for all $0 \leq \epsilon \leq \infty, 1+\epsilon \in [1+\epsilon,\infty]$, and $0 \leq \epsilon \leq \infty$, have

$$\left\|\sum_j\left((1+\epsilon)+i\cdot+A_j\right)^{-1}\right\|_{\mathcal{M}_{(1+\epsilon),(1+\epsilon)}(\mathbb{R};\mathcal{L}(X_{\gamma_j},X))}$$
$$\leq(1+\epsilon)\left((1+\epsilon)_{(1+\epsilon)}(1+\epsilon)^{--\epsilon^2-\epsilon(\beta+1)-\beta-1}+\left(\frac{1+\epsilon}{\epsilon}\right)_{(1+\epsilon)}(1+\epsilon)^{-\left(\frac{1}{1+\epsilon}\right)}\right)$$
$$(1<\epsilon<\infty), \tag{19}$$

where $(1+\epsilon)_{1+\epsilon} = (1+\epsilon)^{-(\epsilon^2+2\epsilon+\beta)-1}\Gamma(\beta+\epsilon+1)^{\left(\frac{1}{1+\epsilon}\right)}$ and $\left(\frac{1+\epsilon}{\epsilon}\right)_{(1+\epsilon)} = (1+\varepsilon)^{-\left(\frac{1}{1+\epsilon}\right)}$ for $1+\epsilon<\infty$, and $(1+\epsilon)_\infty = e^{-(\beta+\epsilon)}(\beta+\epsilon)^{(\beta+\epsilon)}$ and $\left(\frac{1+\epsilon}{\epsilon}\right)_\infty = 1$. Furthermore,

$$sup\left\{\left\|\sum_j\left((1+\epsilon)+i\cdot+A_j\right)^{-1}\right\|_{\mathcal{L}(X_{\gamma_j},X)}\middle|\xi_j\in\mathbb{R}\right\}\leq R_{X_\gamma \ ,X}\left(\left\{\sum_j\left((1+\epsilon)+i\cdot+A_j\right)^{-1}\middle|\xi_j\in\mathbb{R}\right\}\right)$$
$$\leq(1+\epsilon)\left(\Gamma(\beta+\epsilon+1)(1+\epsilon)^{-(\beta+\epsilon)-1}+\tfrac{1}{1+\epsilon}\right). \tag{20}$$

**Proof**. It follows that $\mathbb{C}_- \subseteq \rho(A_j)$. Claim

$$\left\|e^{-(1+\epsilon)}\right\|\ \sum_j T^j(\cdot)\left\|\mathcal{L}\left(X_\gamma \ ,X\right)\right\|_{L^{(1+\epsilon)}(0,\infty)}\leq(1+\epsilon)\left((1+\epsilon)_{(1+\epsilon)}(\beta+\epsilon)^{-\epsilon^2-\epsilon(\beta+1)-\beta-1}+\right.$$

$$\left(\frac{1+\epsilon}{\epsilon}\right)_{(1+\epsilon)}(1+\epsilon)^{-\frac{1}{(1+\epsilon)}}\Big) \leq (1+\epsilon)\left((1+\epsilon)_{1+\epsilon}(1+\epsilon)^{-(\epsilon^2+2\epsilon+\beta)-1}+\left(\frac{1+\epsilon}{\epsilon}\right)_{(1+\epsilon)}(1+\epsilon)^{-\left(\frac{1}{1+\epsilon}\right)}\right)$$

$$(1<\epsilon<\infty). \tag{21}$$

To prove this claim, first peeking $\epsilon<\infty$. Then $\left\|e^{-(1+\epsilon)}\right\| \sum_j T^j(\cdot) \left\|\mathcal{L}(X_\gamma \ ,X)\right\|_{L^{(1+\epsilon)}(0,\infty)}$

$$\leq (1+\epsilon)\left(\int_0^\infty e^{-(1+3\epsilon+2\epsilon^2+\epsilon^3)}\left((1+\epsilon)^{(\beta+\epsilon)}+1\right)^{1+\epsilon}d(1+\epsilon)\right)^{\left(\frac{1}{1+\epsilon}\right)} \leq$$

$$\Bigg(\left(\int_0^\infty e^{-(1+3\epsilon+2\epsilon^2+\epsilon^3)}(1+\epsilon)^{(\epsilon^2+2\epsilon+\beta)-1}d(1+\epsilon)\right)^{\left(\frac{1}{1+\epsilon}\right)}$$

$$+\left(\int_0^\infty e^{-(1+3\epsilon+2\epsilon^2+\epsilon^3)}d(1+)\right)^{\left(\frac{1}{1+\epsilon}\right)}\Bigg) \tag{22}$$

$$\leq (1+\epsilon)\left(\left((\epsilon^2+2\epsilon+1)\ \right)^{-(\epsilon^2+2\epsilon+\beta)-1}\left(\int_0^\infty e^{-(1+\epsilon)}(1+\epsilon)^{\beta+\epsilon}d(1+\epsilon)\right)^{\left(\frac{1}{1+\epsilon}\right)}\right.$$

$$\left.+(\epsilon^2+2\epsilon+1)^{-\left(\frac{1}{1+\epsilon}\right)}\right)$$

$$=(1+\epsilon)\left((1+\epsilon)_{(1+\epsilon)}(1+\epsilon)^{-\epsilon^2-\epsilon(\beta+1)-\beta-1}+\left(\frac{1+\epsilon}{\epsilon}\right)_{(1+\epsilon)}(1+\epsilon)^{-\left(\frac{1}{1+\epsilon}\right)}\right).$$

Second hand, for $1+\epsilon=\infty$ a simple optimization argument shows that

$$\sup_{\epsilon\geq -1}(1+\epsilon)^{-(\epsilon^2+2\epsilon+1)}\left\|\sum_j T^j\,(1+\epsilon)\right\|_{\mathcal{L}(X_{\gamma_j},X)} \leq (1+\epsilon)\left(\sup_{\epsilon\geq -1}(1+\epsilon)^{-(\epsilon^2+2\epsilon+1)}\,(1+\varepsilon)^{(\beta+\epsilon)}+1\right)$$

$$=(1+\epsilon)\left(e^{-(\beta+\epsilon)}(\beta+\epsilon)^{(\beta+\epsilon)}(\beta+\epsilon)^{-(\beta+\epsilon)}+1\right).$$

Set $m_{(1+\epsilon)}(\xi_j):=((1+\epsilon)+i\xi_j+A_j)^{-1}$ for $\epsilon>-1$ and $\xi_j\in\mathbb{R}$. For $1+\epsilon<\infty$ let $f_j\in S(\mathbb{R})\otimes X$, and for $1+\epsilon=\infty$ let $f_j$ be an $X$-valued simple function. Note that $e^{-(1+\epsilon)}\sum_j\left\|T^j(\cdot)\right\|_{\mathcal{L}(X_\gamma \ ,X)}\in L^1(\mathbb{R})$. It hence follows in a straightforward manner (Eisner & Zwart, 2007, Lemma 3.1) that

$$\sum_j((1+\epsilon)+i\xi_j+A_j)^{-1}x^j=\int_0^\infty\sum_j e^{-(1+\epsilon)((1+\epsilon)+i\xi_j)}\,T^j(1+\epsilon)x^j d(1+\epsilon)\,\left(x^j\in X_{\gamma_j},\xi_j\in\mathbb{R}\right)$$

and

$$\sum_j T^j_{m_{(1+\epsilon)}}\left(f_j\right)=\int_0^\infty e^{-(\epsilon^2+2\epsilon+1)}\,\sum_j T^j(1+\epsilon)f(0)d(1+\epsilon) \qquad ((1+\epsilon)\in\mathbb{R}).$$

The latter equality, (20) and Young's inequality for sequence of operators-valued kernels (Arendt, Batty, Hieber, & Neubrander, 2011, Proposition 1.3.5) yield (19). On the other hand, applying and (22) with $\epsilon=0$ to $(1+\epsilon)\mapsto e^{-(\epsilon^2+2\epsilon+1)}\,T^j(1+\epsilon)$ yields (20).

For $-A_j$ a standard $n\times n$ Jordan block acting on $X=\mathbb{R}^n, n\geq 2$, there exists $\epsilon\geq -1$ such that

$$\left(\frac{1}{1+\epsilon}\right)((1+\epsilon)^{n-1}+1) \le \left\|\sum_j T^j\,(1+\epsilon)\right\|_{\mathcal{L}(X)} \le (1+\epsilon)((1+\epsilon)^{n-1}+1) \qquad (\varepsilon \ge -1)$$

and

$$\left\|\sum_j \left((1+\epsilon)+i\xi_j+A_j\right)^{-1}\right\|_{\mathcal{L}(X)} \le \left\|\sum_j \left((1+\epsilon)+A_j\right)^{-1}\right\| \le (1+\epsilon)((1+\epsilon)^{-n}+(1+\epsilon)^{-1})$$

$$(\epsilon > -1, \xi_j \in \mathbb{R}).$$

This shows that (20) is optimal. Note that in this case $R$-boundedness and uniform boundedness simultaneity since $X$ is a Hilbert space.

**Remark 3.12**. One might be tempted to think that the more restrictive R-bounded analogue of (2) which seems in (20), means

$$R_X\left(\sum_j \left\{\left((1+\epsilon)+i\xi_j+A_j\right)^{-1}\middle|\xi_j \in R\right\}\right) \le \sum_j g^j\,(1/(1+\epsilon)) \qquad (1<\epsilon<\infty),$$

can be used to extend the conclusion of Theorem 1.1 to more general Banach spaces. However, the example at the end of Section 3.4 shows that this is not the case for certain positive semigroupson $L^{(1+\epsilon)}(\Omega) \cap L^{(1+\epsilon)}(\Omega)$, for $\Omega$ a measure space.

**Corollary 3.13**. Let $-A_j$ be the sequence of generators of a $C_0$-semigroup $(T^j\,(1+\epsilon))_{\epsilon\ge -1}$ on a Banachspace $X$ such that $\mathbb{C}_- \subseteq \rho(A_j)$, and let $\beta+\epsilon, \gamma \in [0,\infty)$. Then the following conditions are equivalent:

(1) there exists $\epsilon \ge -1$ such that $\left\|\sum_j T^j\,(1+\epsilon)\right\|_{\mathcal{L}(X_\gamma\ ,X)} \le (1+\epsilon)((1+\varepsilon)^{(\beta+\epsilon)}+1)$ for all $\varepsilon \ge -1$;

(2) there exist $0 \le \epsilon \le \infty$ and a $\frac{1+\epsilon}{\epsilon} \ge 0$ such that

$$\sum_j \left((1+\epsilon)+i\cdot+A_j\right)^{-1} \mathcal{M}_{(1+\epsilon),(1+\epsilon)}\left(\mathbb{R};\mathcal{L}\left(X_\gamma\ ,X\right)\right) \le \frac{1+\epsilon}{\epsilon}\left((1+\varepsilon)^{-(\beta+\epsilon)}+1\right) \quad (1<\epsilon<\infty). \tag{23}$$

**Proof**. Theorem 3.2 contains (2)⇒(1), and (1)⇒(2) follows from Theorem 3.11 by letting $\epsilon = 0$ and $\epsilon = \infty$.

Note that Corollary 3.13 also characterizes semigroups which grow sublinearly, and in specific uniformly bounded semigroups. To characterize such semigroups it would not be possible to replace the multiplier norm in (23) by a supremum series norms, since $\left\|R\sum_j(\lambda_j, A_j)\right\|_{\mathcal{L}(X)} \ge dist\sum_j(\lambda_j, \sigma(A_j))^{-1}$ for all $\lambda_j \in \rho(A_j)$.

**Note**. We can deduce that:

$$\left((1+\epsilon)^{-(\beta+\epsilon)}+1\right) \le \frac{\epsilon}{1+\epsilon}\sum_j g^j\,(1/(1+\epsilon)).$$

From (9) and (23).

*3.7 Auxiliary Results*

The theorems in this article also apply if $A_j$ is $n \times n$ matrix acting on $X = \mathbb{R}^n$, $n \in \mathbb{N}$. For example, if

$$\left\|\sum_j ((1+\epsilon) + i\xi_j + A_j)^{-1}\right\|_{\mathcal{L}(X)} \leq \sum_j g^j\,(1/(1+\epsilon)) \qquad (\epsilon > -1, \xi_j \in \mathbb{R})$$

then one obtains $\left\|\sum_j e^{-(1+\epsilon)A_j}\right\|_{\mathcal{L}(X)} \leq e\sum_j M_{\omega^j}\,(1+2M\omega^j)(g^j(1+\epsilon)+1)$ for all $\varepsilon > -1$ if $\mathbb{R}^n$ isendowed with the standard norm, or if $(e^{-(1+\epsilon)A_j})_{\epsilon \geq -1}$ is positive and $\mathbb{R}^n$ is endowedwith the $\ell_{(1+\epsilon)}$-norm, $0 \leq \epsilon \leq \infty$. Here $\omega^j, M$and $M_{\omega^j}$ are as in (3). Note that this series estimate does not depend on n but that it does require knowledge of $\omega^j, M$ and $M_{\omega^j}$. If these constants are unknown then the argument used yields the following statement, which is presumably well known to experts. For the convenience of the include the proof. Recall that it suffices to consider the case where g grows at least linearly at infinity and $g^j(1+\epsilon) = O(1+\epsilon)$ as $1+\epsilon \to 0$.

**Proposition 3.14**. Let $X$ be an n-dimensional normed vector space, $n \in \mathbb{N}$, and let $A_j \in \mathcal{L}(X)$ be such that $\mathbb{C}_- \subseteq \rho(A_j)$. Assume that there exists a non decreasing $g^j: (0,\infty) \to (0,\infty)$ such that

$$\left\|\sum_j ((1+\epsilon) + i\xi_j + A_j)^{-1}\right\|_{\mathcal{L}(X)} \leq \sum_j g^j\,(1/(1+\epsilon)) \qquad (1 < \epsilon < \infty, \xi_j \in \mathbb{R}).$$

Therefore $\left\|e^{-(1+\epsilon)A_j}\right\|_{\mathcal{L}(X)} \leq en\frac{g^j(1+\epsilon)}{(1+\epsilon)}$ for all $\epsilon > -1$.

**Proof**. Let a, $\varepsilon > -1$ and write, as in the proof of Proposition 3.1,

$$e^{-(1+\epsilon)(1+\epsilon)}\sum_j T^j(1+\epsilon) = \frac{1}{2\pi i}\int_{i\mathbb{R}}\sum_j e^{-z^j(1+\epsilon)}\,R\left(z^j, A_j + (1+\epsilon)\right)dz^j.$$

Let $F \in \mathcal{L}(X)^*$be such that $\|F\|_{\mathcal{L}(X)^*} \leq 1$ and $F\sum_j(T^j(1+\epsilon)) = \sum_j\left\|T^j(1+\epsilon)\right\|_{\mathcal{L}(X)}$. Integrationby parts yields

$$\begin{aligned} e^{-(\epsilon^2+2\epsilon+1)}\sum_j\left\|T^j(1+\epsilon)\right\|_{\mathcal{L}(X)} &= \frac{1}{2\pi i}\int_{i\mathbb{R}}\sum_j e^{-z^j(1+\epsilon)}\,F\left(R\left(z^j, A_j + (1+\epsilon)\right)\right)dz^j \\ &= \frac{1}{2\pi i(1+\varepsilon)}\int_{i\mathbb{R}}\sum_j e^{-z^j(1+\epsilon)}\,F\left(R\left(z^j, A_j + (1+\epsilon)\right)\right)'\,dz^j. \end{aligned}$$

Sees that $z^j \mapsto F(R(z^j, A_j + (1+\epsilon)))$ is a rational scalar-valued map with numerator and denominator of degree at most $n$. shows that

$$e^{-(\epsilon^2+2\epsilon+1)}\sum_j\left\|T^j(1+\epsilon)\right\|_{\mathcal{L}(X)} \leq \frac{n}{(1+\epsilon)}\sup_{z^j \in i\mathbb{R}}\left|F\sum_j\left(R\left(z^j, A_j + (1+\epsilon)\right)\right)\right| \leq \frac{n\sum_j g^j\,(1/(1+\epsilon))}{(1+\epsilon)}.$$

Finally, set $\epsilon^2 + 2\epsilon$ to conclude the proof.

Proposition 3.14 is sharp in the case where $g^j(1+\epsilon) = (\epsilon^2 + 2\epsilon + 1)$ for some $\epsilon \geq -1$ and all $\epsilon > -1$.

Finally, as a corollary of Theorem 3.6 that $V(A_j) := (1-A_j)(1+A_j)^{-1}$ of a semigroup sequence of generators $-A_j$ on a Banach space $X$ with $-1 \in \rho(A_j)$. Recall that each eventually differentiable semigroup, and in particular each analytic semigroup, is asymptotically analytic. Also, if $-A_j$ the sequence of generates a $C_0$-semigroup $(T^j(1+\epsilon))_{\epsilon\geq -1}$ on aHilbert space $X$ such that $(1+\epsilon)_0^{\infty}(-A_j) < 0$, then $(T^j(1+\epsilon))_{\epsilon\geq -1}$ are asymptotically analytic. Hence the following result both extends and improves.

**Corollary 3.15**. Let $(T^j(1+\epsilon))_{\epsilon\geq -1}$ be an asymptotically analytic $C_0$-semigroup with sequence of generators $-A_j$ on a Banach space $X$ such that $-1 \in \rho(A_j)$. Assume that there exist $k \in \mathbb{N}_0$ and $\epsilon \geq -1$ such that

$$\left\| V \sum_j (A_j)^n \right\|_{\mathcal{L}(X)} \leq (1+\epsilon)\, n^k (n \in \mathbb{N}).$$

Then there exists $\frac{1+\epsilon}{\epsilon} \geq 0$ such that $\left\|\sum_j T^j(1+\epsilon)\right\|_{\mathcal{L}(X)} \leq \frac{1+\epsilon}{\epsilon}(1+(1+\epsilon)^{k+1})$ for all $\epsilon \geq -1$.

**Proof**. First note that $S_0^{\infty}(-A_j) < 0$, since $(T^j(1+\varepsilon))_{\epsilon\geq -1}$ is asymptotically analytic, then show that

$$\left\|((1+\epsilon)+i\xi_j+A_j)^{-1}\right\|_{\mathcal{L}(X)} \leq (1+\varepsilon)_1(1+\epsilon)^{-k-1} \qquad (\epsilon > -1, \xi_j \in \mathbb{R})$$

for some $(1+\varepsilon)_1 \geq 0$. Theorem 3.6 then concludes the proof.

## 4. Application to a Perturbed Wave Equation

Constructed $C_0$-semigroup $(T^j(1+\epsilon))_{\epsilon\geq -1}$ with the sequence of generators $-A_j$ on a Hilbert space such that $\omega_0^j(T^j) > (1+\epsilon)(-A_j)$. Onemight be tempted to think that this phenomenon only happens in rather academic situations. More exactly, the $C_0$-group $(T^j(1+\epsilon))_{1+\epsilon\geq\mathbb{R}}$ with generator $-A_j$ which arises when formulating this wave equation as an abstract Cauchy problem has the property that $(1+\epsilon)(-A_j) = 0 = (1+\varepsilon)(A_j)$but $\omega_0^j(T^j) \geq \frac{1}{2}$. In prove that $\omega_0^j(T^j) = \frac{1}{2}$, a matter which was left open. In fact, Theorem 4.1 below yields a more precise growth bound for $(T^j(1+\varepsilon))_{1+\epsilon\geq\mathbb{R}}$.

On the two-dimensional torus $\mathbb{T}^2 := [0,2\pi]^2$, under the usual identification modulo $2\pi$, consider

$$\begin{cases} (u^j)_{(\epsilon^2+2\epsilon+1)} = (u^j)_{x^jx^j} + (u^j)_{y^jy^j} + e^{iy^j}(u^j)_{x^j}, \quad 1<\epsilon<\infty, x^j, y^j \in \mathbb{T}, \\ (u^j)(0,x^j,y^j) = f_j(x^j,y^j), (u^j)_{(1+\epsilon)}(0,x^j,y^j) = g^j(x^j,y^j), x^j, y^j \in \mathbb{T}, \end{cases} \tag{24}$$

for $f_j, g^j \in L^2\mathbb{T}^2$. For $(1+\epsilon) \in \mathbb{R}$ let $H^{(1+\epsilon)}\mathbb{T}^2 = W^{2,(1+\epsilon)}\,\mathbb{T}^2$ be the second order Sobolev space equipped with the following convenient series norms:

$$\sum_j \|f_j\|_{H^{(1+\epsilon)}\mathbb{T}^2} = \left(\sum_j \left|\hat{f}_j(0)\right|^2 + \sum_{k\in\mathbb{Z}^2\setminus\{0\}} |k|^{2(1+\varepsilon)} \left|\hat{f}_j(k)\right|^2\right)^{1/2} \quad \left(f_j \in H^{(1+\epsilon)}\mathbb{T}^2\right).$$

Clearly, this series norm is equivalent to the standard norm on $H^{(1+\epsilon)}\mathbb{T}^2$:

$$\left\|\sum_j f_j\right\|_{H^{(1+\epsilon)}\mathbb{T}^2} \leq \left(\sum_{k\in\mathbb{Z}^2}(1+|k|^2)^{(1+\epsilon)} \left|\hat{f}_j(k)\right|^2\right)^{1/2} \leq C_{(1+\epsilon)}\left\|\sum_j f_j\right\|_{H^{(1+\epsilon)}(\mathbb{T}^2)} \tag{25}$$

for some $C_{(1+\varepsilon)} \geq 0$ and all $f_j \in H^{(1+\epsilon)}(\mathbb{T}^2)$. Then (24) can be formulated as an abstract Cauchy problem on the Hilbert space $X:=H^1\mathbb{T}^2\times L^2\mathbb{T}^2$:

$$\frac{d}{d(1+\epsilon)}\binom{u^j}{v^j}+A_j\binom{u^j}{v^j}=0 \tag{26}$$

and $(u^j)(0),v^j(0))=(f_j,g^j)$, where $A_j=\left(A_j\right)_0+B_j$ with $D(A_j)=H^2\mathbb{T}^2\times H^1\mathbb{T}^2$,

$$\left(A_j\right)_0=\begin{pmatrix}0 & -1\\ -\Delta & 0\end{pmatrix} \quad and \quad B_j=\begin{pmatrix}0 & 0\\ -M\dfrac{\partial}{\partial x^j} & 0\end{pmatrix}.$$

Here $\Delta$ is the Laplacian with $D(\Delta)=H^2\mathbb{T}^2$, and $M: L^2\mathbb{T}^2\to L^2\mathbb{T}^2$ is given by $Mf_j(x^j,y^j)=e^{iy^j}f_j(x^j,y^j)$ for $f_j\in L^2\mathbb{T}^2$ and $x^j,y^j\in\mathbb{T}$. Using Fourier seriesone easily checks that $-\left(A_j\right)_0$ generates a $C_0$-group. More exactly, let $e_k(x^j,y^j):=(2\pi)^{-1}e^{ik(x^j,y^j)\,\cdot}$ for $k\in\mathbb{Z}^2$. Taking the discrete Fourier tranform, the system can be solved explicitly.

$$\frac{d}{d(1+\varepsilon)}\begin{pmatrix}\varphi_j\\ \psi_j\end{pmatrix}+\left(A_j\right)_0\begin{pmatrix}\varphi_j\\ \psi_j\end{pmatrix}=0$$

Let $h_k:=\frac{1}{2\pi}\int_{\mathbb{T}^2}e^{-ik\cdot(x^j,y^j)}\,h(x^j,y^j)dx^jdy^j, k\in\mathbb{Z}^2$ be the Fourier coefficients of $h\in L^2(\mathbb{T}^2)$. Then

$$\sum_j\psi_j\,(1+\epsilon)=\sum_j\left((f_j)_0+(1+\epsilon)g_0^j\right)e_0+\sum_{k\in\mathbb{Z}^2\setminus\{0\}}\left(cos(|k|1+\epsilon)\left(f_j\right)_k+\frac{sin(|k|(1+\epsilon))}{|k|}g_k^j\right)e_k,$$

For $1+\epsilon\in\mathbb{R}$. Set $e^{-(1+\epsilon)\left(A_j\right)_0}\binom{f_j}{g^j}:=\binom{\varphi_j(1+\epsilon)}{\psi_j(1+\epsilon)}$. one has

$$\sum_j\left\|\left(\varphi_j(1+\varepsilon),\psi_j(1+\varepsilon)\right)\right\|_X^2$$

$$=\sum_j\left|\left(f_j\right)_0+(1+\epsilon)g_0^j\right|^2+\sum_j\left|g_0^j\right|^2+\sum_{k\in\mathbb{Z}^2\setminus\{0\}}\left(|k|^2\left|\left(f_j\right)_k\right|^2+\left|g_k^j\right|^2\right),$$

$$\leq 2\sum_j\left|\left(f_j\right)_0\right|^2+\sum_{k\in\mathbb{Z}^2\setminus\{0\}}|k|^2\sum_j\left|\left(f_j\right)_k\right|^2+2\sum_j\left|(1+\epsilon)g_0^j\right|^2+\sum_j\left|g_0^j\right|^2$$
$$+\sum_j\sum_{k\in\mathbb{Z}^2\setminus\{0\}}\left|g_k^j\right|^2$$

$$\leq 2\left\|\sum_j f_j\right\|_{H_1(\mathbb{H}^2)}^2+(1+2(1+\epsilon)^2)\left\|\sum_j g^j\right\|_{H_1(\mathbb{H}^2)}^2$$

$$\leq 2(1+|1+\epsilon|)^2\sum_j\left\|\left(f_j,g^j\right)\right\|_X^2,$$

so that $\left\|\sum_j e^{-(1+\varepsilon)\left(A_j\right)_0}\right\|_{\mathcal{L}(X)}\leq\ \sqrt{2}\ (1+|1+\epsilon|)$ for all $1+\epsilon\in\mathbb{R}$. One could alternatively get a norm estimate using Theorem 3.4, but in this case one obtains only a quadratic bound.

Since $\left\|\sum_j B_j\right\|_{\mathcal{L}(X)}\leq 1$ , standard perturbation theory shows that $-A_j=-(A_j)_0-B_j\ the\ sequence\ of$ generates $C_0$-group $(T^j(1+\varepsilon))_{1+\epsilon\,\in\mathbb{R}}$ with

$$\left\|\sum_j T^j\,(1+\epsilon)\right\|_{\mathcal{L}(X)} \leq \sqrt{2}\, e^{(1+\sqrt{2})|1+\epsilon|} \qquad (1+\epsilon \in \mathbb{R}). \tag{27}$$

It was shown in Jan and Mark (2018) that $\sigma(A_j) \subseteq i\mathbb{R}$ and $\omega_0^j(T^j) \geq \frac{1}{2}$, and by thesame method one sees that $\sum_j \omega_0^j\,(S) \geq \frac{1}{2}$ for $(S(1+\epsilon))_{\epsilon\geq -1} := \sum_j (T^j(1+\epsilon)^{-1})_{\epsilon\geq -1}$, the semigroup sequence generated by $A_j$. The next theorem is the main result of this section. It shows thatthese lower bounds are optimal and in doing so significantly improves (27).

**Theorem 4.1**. Let $X$ and $A_j$ be as before, and let $(T^j(1+\varepsilon))_{1+\epsilon\in\mathbb{R}}$ and $(S(1+\varepsilon))_{1+\epsilon\in\mathbb{R}}$ be the $C_0$ -semigroups generated by $-A_j$ and $A_j$, respectively. Then $\sum_j \omega_0^j\,(T^j) = \sum_j \omega_0^j\,(S) = \frac{1}{2}$. Furthermore, there exists $\epsilon \geq -1$ such that

$$\left\|\sum_j T^j\,(1+\epsilon)\right\|_{\mathcal{L}(X)} \leq (1+\epsilon)(1+|1+\epsilon|)e^{|1+\epsilon|/2} \qquad (1+\varepsilon \in \mathbb{R}).$$

**Note.** Deduce that:

$$\frac{\sqrt{2}e^{(1+\sqrt{2})|1+\epsilon|}}{(1+\epsilon)(1+|1+\epsilon|)} \leq e^{|1+\epsilon|/2}$$

**Proof**. From (27) and Theorem (4.1)

**Remark 4.2**. For each $\epsilon \geq -1$ there exists a $(1+\epsilon)_{(1+\epsilon)} \geq 0$ such that $\left\|\sum_j (\frac{1}{2} + i\xi_j \pm A_j)^{-1}\right\|_{\mathcal{L}(X)} \leq (1+\epsilon)_{(1+\epsilon)}$ for $|\xi_j| \leq (1+\epsilon)$, since $\sigma(A_j) \subseteq i\mathbb{R}$, and it follows from Theorem 4.1 that $(1+\epsilon)_{(1+\epsilon)} \to \infty$ as $(1+\epsilon) \to \infty$. It would be interesting to study the asymptotic behavior of $\sum_j \left\|(\frac{1}{2} + i\xi_j \pm A_j)^{-1}\right\|_{\mathcal{L}(X)}$ as $|\xi_j| \to \infty$. Moreover, if $\sum_j \left\|e^{-|1+\varepsilon|/2} T^j(1+\epsilon)\right\|_{\mathcal{L}(X)}$ were to grow asymptotically linearly as $1+\varepsilon \to \infty$ then this would solve the optimality issueleft open after Theorem 3.4

The proof of Theorem 4.1 relies on two lemmas. The first collects some basic series estimates.

**Lemma 4.3**. Let $z^j \in \mathbb{C}$ be such that $|R\sum_j e(z^j)\,| \geq \frac{1}{2}$, and let $y^j \in \mathbb{R}$. Then (i) $\sum_j \frac{|z^j|^2}{|(z^j)^2 + (y^j)^2|^2} \leq 4$, (ii) $\sum_j \frac{(y^j)^2+1}{|(z^j)^2+(y^j)^2|^2} \leq 16$, (iii) $\sum_j \frac{|z|^4}{|(z^j)^2+(y^j)^2|^2} \leq 32(\sum_j (y^j)^2 + 1)$.

**Proof**. Write $z^j = (1+\epsilon)$ +i$(1+\epsilon)$ for $(1+\epsilon), (1+\epsilon) \in \mathbb{R}$ with $|(1+\epsilon)| \geq 1/2$. Then (i) and (ii) follow from

$$\begin{aligned}\sum_j |(z^j)^2 + (y^j)^2|^2 \\ &= \sum_j ((y^j)^2 - (1+\varepsilon)^2)^2 + ((1+\epsilon))^4 + 2\sum_j (y^j)^2\,((1+\epsilon))^2 \\ &+ 2((1+\epsilon))^2(1+\varepsilon)^2 \geq max \sum_j \left(\frac{1}{16}(1+(y^j)^2), \frac{1}{4}\left|z^j\right|^2\right).\end{aligned}$$

For (iii) note that

$$\sum_j |z^j|^4 \leq \sum_j \left(\left|(z^j)^2 + (y^j)^2\right| + (y^j)^2\right)^2 \leq$$

$$2\sum_j |(z^j)^2 + (y^j)^2|^2 + 2\sum_j (y^j)^4 ,$$

divide by $\sum_j |(z^j)^2 + (y^j)^2|^2$, and use (ii).

The following lemma contains the required resolvent estimates for $A_j$.

**Lemma 4.4**. Let $X$and $A_j$ be as before. Then there exists $\epsilon \geq -1$ such that for all $\epsilon > 0, \xi_j \in \mathbb{R}$ and $\sum_j \lambda_j = \pm\left(\frac{1}{2} + \epsilon\right) + i\sum_j \xi_j$ one has

$$\left\|\sum_j (\lambda_j + A_j)^{-1}\right\|_{\mathcal{L}(X)} \leq (1+\epsilon)max(\epsilon^{-1}, 1).$$

**Proof**. Let $\lambda_j \in \mathbb{C}\backslash i\mathbb{R}$, $(u^j, v^j) \in D(A_j)$ and $\left(f_j, g^j\right) \in X$ be such that $(\lambda_j + A_j)(u^j, v^j) = (f_j, g^j)$. Then

$$\sum_j \lambda_j^2 \ u^j - \Delta u^j - \sum_j e^{iy^j} (u^j)_{x^j} = \sum_j g^j + \sum_j \lambda_j f_j \tag{28}$$

in $L^2(\mathbb{T}^2)$. Since $v^j = \sum_j \lambda_j u^j - \sum_j f_j$, it suffices to prove

$$\sum_j \left\|u^j\right\|_{H^1\mathbb{T}^2} + \sum_j \left\|\lambda_j u^j\right\|_{L^2\mathbb{T}^2} \leq (1+\epsilon)max(1, \epsilon^{-1})(\sum_j \left\|f_j\right\|_{H^1\mathbb{T}^2} + \sum_j \left\|g^j\right\|_{L^2\mathbb{T}^2} \tag{29}$$

if $\lambda_j = \pm\left(\frac{1}{2} + \epsilon\right) + i\xi_j$ for $\epsilon > 0$ and $\xi_j \in \mathbb{R}$. Write $\sum_j u^j = \sum_{(m,n)\in\mathbb{Z}^2} (1+\epsilon)_{m,n} e_{m,n}$ with $((1+\epsilon)_{m,n})_{m,n\in\mathbb{Z}}$ the Fourier coefficients of $(u^j)$ and $(e_{m,n})_{m,n\in\mathbb{Z}}$ the normalized trigonometric basis of $L^2(\mathbb{T}^j)^2$. Then (28) yields

$$\sum_j (\lambda_j^2 + m^2 + n^2)_{(u^j)_{m,n}} = im(1+\varepsilon)_{m,n-1} + \sum_j g_{m,n}^j + \sum_j \lambda_j \left(f_j\right)_{(m,n)} \qquad (m, n \in \mathbb{Z}).$$

Using that $4|1+\epsilon|^2 \leq (1+\delta)|1+\epsilon|^2 + (1+\delta^{-1})|1+\epsilon|^2$ for any fixed $\delta > 0$ and all $(1+\epsilon), (1+\epsilon) \in \mathbb{C}$, one has

$$\left|\sum_j (u^j)_{m,n}\right|^2 \leq \sum_j \frac{(1+\delta)\left|m(u^j)_{m,n-1}\right|^2}{\left|\sum_j \lambda_j^2 + m^2 + n^2\right|^2} + \left(1 + \frac{1}{\delta}\right)\sum_j \left(\frac{\left|g_{m,n}^j\right|}{\left|\lambda_j^2 + m^2 + n^2\right|} + \frac{\left|\lambda_j (f_j)_{m,n}\right|}{\left|\lambda_j^2 + m^2 + n^2\right|}\right)^2 \tag{30}$$

First bound $\left\|u^j\right\|_{H^1(\mathbb{T}^2)}$ in (29). From (30) obtain

$$\sum_{m,n\in\mathbb{Z}} (m^2 + n^2 + 1)\left|\sum_j (u^j)_{m,n}\right|^2$$

$$\leq (1+\delta)\sum_{m,n\in\mathbb{Z}} \sum_j \frac{m^2(m^2 + (n+1)^2 + 1)\left|(u^j)_{m,n}\right|^2}{\left|\sum_j \lambda_j^2 + m^2 + (n+1)^2\right|^2} + \sum_j (1+\epsilon)_{f_j, g^j}^2$$

for

$$\sum_j (1+\epsilon)_{f_j, g^j}^2 = \left(1 + \frac{1}{\delta}\right)\sum_j \sum_{k\in\mathbb{Z}^2} \left(\frac{(|k|^2+1)^{1/2}\left|g_k^j\right|}{\left|\lambda_j^2 + |k|^2\right|} + \frac{(|k|^2+1)^{1/2}\left|\lambda_j \left(f_j\right)_k\right|}{\left|\lambda_j^2 + |k|^2\right|}\right)^2.$$

Lemma 4.3 (i) and (ii) yield a $(1+\epsilon)_1 \geq 0$ such that $\sum_j (1+\epsilon)_{f_j,g^j} \leq (1+\epsilon)_1(1+\delta^{-1})1/2\sum_j(\|f_j\|_{H^1} + \|g^j\|_{L^2})$, so that

$$\sum_{m,n\in\mathbb{Z}}(m^2+n^2+1)\left|\sum_j (u^j)_{m,n}\right|^2 \left(1-(1+\delta)(y^j)_{m,n}\right) \leq (1+\epsilon)_1^2(1+\delta^{-1})\sum_j\left(\|f_j\|_{H^1} + \|g^j\|_{L^2}\right)^2 \quad (31)$$

for

$$\sum_j (y^j)_{m,n} := \frac{m^2(m^2+(n+1)^2+1)}{(m^2+n^2+1)\sum_j|\lambda_j^2+m^2+(n+1)^2|^2} \qquad (m,n\in\mathbb{Z}).$$

Assume that $\lambda_j = (1+\varepsilon)+i\xi_j$ for $\xi_j\in\mathbb{R}$ and $|(1+\varepsilon)| > \frac{1}{2}$. Then a simple minimization argument yields

$$\sum_j|\lambda_j^2+m^2+(n+1)^2|^2 = \sum_j\left((1+\varepsilon)^2-{\xi_j}^2+m^2+(n+1)^2\right)^2 + 4\sum_j(1+\varepsilon)^2{\xi_j}^2 \geq 4(1+\varepsilon)^2(+m^2+(n+1)^2), \quad (32)$$

from which it follows that $\sum_j (y^j)_{m,n} \leq \frac{1}{4(1+\varepsilon)^2}$ for all $m,n\in\mathbb{Z}$. Collecting this with (25) and (4.8), obtain that for $\delta\in(0,4(1+\varepsilon)^2-1)$ one has

$$\left\|\sum_j u^j\right\|_{H^1\mathbb{T}^2} \leq (1+\epsilon)_1\frac{2|1+\epsilon|(1+\delta^{-1})^{1/2}}{\left(4(1+\epsilon)^2-(1+\delta)\right)^{1/2}}\left(\left\|\sum_j f_j\right\|_{H^1\mathbb{T}^2} + \left\|\sum_j g^j\right\|_{L^2\mathbb{T}^2}\right).$$

For $\mathbb{T}^2 > 0$ such that $|(1+\varepsilon)| = \frac{1}{2}+\epsilon$ one obtains $(1+\epsilon)_2 \geq 0$ independent of $\epsilon$ such that

$$\left\|\sum_j u^j\right\|_{H^1\mathbb{T}^2} \leq (1+\epsilon)_2 max(1,\varepsilon^{-1})\left(\left\|\sum_j f_j\right\|_{H^1\mathbb{T}^2} + \left\|\sum_j g^j\right\|_{L^2\mathbb{T}^2}\right).$$

Bound $\left\|(\lambda_j)_{u^j}\right\|_{L^2(\mathbb{T}^j)^2}$ in 29). From (30) one yields

$$\sum_{m,n\in\mathbb{Z}}\left|\sum_j\lambda_j\right|^2\left|(u^j)_{m,n}\right|^2 \leq (1+\delta)\sum_j\sum_{m,n}\frac{|\lambda_j|^2m^2\left|(u^j)_{m,n}\right|^2}{\left|\lambda_j^2+m^2+(n+1)^2\right|^2} + \sum_j K_{f_j,g^j}^2 \quad (33)$$

where

$$\sum_j (K)_{f_j,g^j}^2 = \left(1+\frac{1}{\delta}\right)\sum_j\sum_{k\in\mathbb{Z}^2}\left(\frac{|\lambda_j||g_k^j|}{|\lambda_j^2+|k|^2|} + \frac{|\lambda_j|^2|(f_j)_k|}{|\lambda_j^2+|k|^2|}\right)^2$$

$$\leq (1+\epsilon)_3(1+\delta^{-1})^{1/2}\left(\left\|\sum_j f_j\right\|_{H^1} + \left\|\sum_j g^j\right\|_{L^2}\right)$$

for some $(1+\epsilon)_3 \geq 0$ by Lemma 4.3 (i) and (iii). Then (33) implies

$$\sum_j\left|\sum_j\lambda_j\right|^2\sum_{m,n}\left|(u^j)_{m,n}\right|^2\left[1-(1+\delta)(z^j)_{m,n}\right] \leq (1+\epsilon)_3^2(1+\delta^{-1})\sum_j\left(\|g^j\|_{L^2} + \|f_j\|_{H^1}\right)^2$$

where

$$\sum_j (z^j)_{m,n} := \frac{m^2}{\sum_j \left|\lambda_j^2 + m^2 + (n+1)^2\right|^2} \leq \frac{1}{4(1+\varepsilon)^2} (m, n \in \mathbb{Z})$$

by (32). As in the previous step this yields a constant $(1+\epsilon)_4 \geq 0$ such that, for $\varepsilon > 0$ such that $|a| = 1$

$$\left\|\sum_j \lambda_j \ (u^j)\right\|_{H^1(\mathbb{T}^2)} \leq (1+\epsilon)_4 max(1, \varepsilon^{-1}) \sum_j \left(\|f_j\|_{H^1(\mathbb{T}^2)} + \|g^j\|_{L^2(\mathbb{T}^2)}\right).$$

This completes the proof of (29).

**Proof of Theorem 4.1**. The inequalities $\sum_j \omega_0^j \ (T^j) \geq \frac{1}{2}$ and $\sum_j (S)\omega_0^j \geq \frac{1}{2}$ shows that the sequence of operators $\frac{-1}{2} + A_j$ and $\frac{-1}{2} - A_j$ satisfy the conditions of Theorem 3.4 with $\sum_j g^j(1+\epsilon) = max(1/1+\epsilon, 1)$ for $\epsilon > -1$, and the latter theorem concludesthe proof.

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