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Original Paper

Lagrange's Mean Value Theorem and Taylor's Theorem and Their Applications

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Abstract

Lagrange's mean value theorem and Taylor's theorem are two important and widely used formulas in calculus courses. In this paper, we introduce the method for proving Lagrange's mean value theorem and Taylor's theorem using Rolle's theorem, and the application of these two theorems in estimating the value of integrals, determining the concavity and convexity of functions, and solving the limits of functions.

Keywords

Lagrange's mean value theorem, Taylor's theorem, concavity of functions

1. Introduction

In the field of mathematical analysis, Lagrange's mean value theorem and Taylor's theorem are important theorems that are closely related to each other. Specifically, Lagrange's mean value theorem can be regarded as a case of Taylor's theorem at. When in Taylor's theorem takes 1, the Taylor polynomial degenerates into a linear function, at which point Taylor's theorem transforms into the form of Lagrange's mean value theorem.

Lagrange's mean value theorem and Taylor's theorem have a wide range of applications in numerical computations (Jiang, 2020), function approximations (Li, 2022; Ma, 2024), and differential equations (Xu, 1998; Fang, 2020). In this paper, we introduce a method for proving Lagrange's mean value theorem and Taylor's theorem by using Rolle's theorem and provide some applications of these two theorems in numerical estimation, determination of the convexity of a function, and solution of the extreme value of a function.

2. Lagrange's Mean Value Theorem and Taylor's Theorem

Theorem 2.1 (**Rolle's theorem**) If $g(x)$ is derivable in (c,d) and continuous in $[c,d]$, and $g(c)=g(d)$, then there exists $\xi \in (c,d)$ such that $g'(\xi)=0$.

In the following we will give a unified method for proving Lagrange's mean value theorem and Taylor's theorem using Theorem 2.1.

Theorem 2.2 (**Lagrange's mean value Theorem**) If $g(x)$ is derivable in (a,b) and continuous in $[a,b]$, then there is $\xi \in (a,b)$ which makes $g'(\xi)=\frac{g(b)-g(a)}{b-a}$.

Proof. Let $G(x)=g(x)-\frac{g(b)-g(a)}{b-a}x$. Obviously, $g(x)$ is derivable in (a,b) and continuous in $[a,b]$, and $G(a)=G(b)=\frac{bg(a)-ag(\mathrm{b})}{b-a}$. Hence by Rolle's theorem, there exists $\xi \in (a,b)$ such that $G'(\xi)=0$, which follows that $g'(\xi)=\frac{g(b)-g(a)}{b-a}$. ■

Theorem 2.3 [**Taylor's theorem**] If $f(x)$ has $n+1$ continuous derivatives on some interval (a,b) containing x_0, then for every $c \in (a,b)$,

$$f(c)=\sum_{k=0}^{n}\frac{f^{(\mathrm{k})}(x_0)}{k!}(c-x_0)^k+\frac{f^{(n+1)}(\xi)}{(n+1)!}(c-x_0)^{n+1},$$

where ξ is between c and x_0.

Proof. Denote $h=f(c)-\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(c-x_0)^k$. Let

$$g(x)=f(x)-\sum_{k=0}^{n}\frac{f^{(\mathrm{k})}(x_0)}{k!}(x-x_0)^k-\frac{h}{(c-x_0)^{n+1}}(x-x_0)^{n+1}.$$

Then $g(x)$ has $n+1$ continuous derivatives on (a,b) and

$$g^{(t)}(x_0)=f^{(t)}(x_0)-f^{(t)}(x_0)=0$$

for $1\le t\le n$. Note that $g(x_0)=0$ and $g(c)=h-h=0$. Hence, there is $\xi_0 \in (a,b)$ such that $g'(\xi_0)=0$ by Rolle's theorem. Then $g'(\xi_0)=g'(x_0)=0$. Applying Rolle's theorem successively, we can get $g^{(n+1)}(\xi)=0$, where $\xi \in (a,b)$. That is

$$g^{(n+1)}(\xi)=f^{(n+1)}(\xi)-\frac{h(n+1)!}{(c-x_0)^{n+1}}=0.$$

Hence,

$$f(c)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(c-x_0)^k+\frac{f^{(n+1)}(\xi)}{(n+1)!}(c-x_0)^{n+1}.$$

This completes the proof. ■

3. Applications of Lagrange's Mean Value Theorem and Taylor's Theorem

3.1 Numerical Estimate

Example 3.1 Estimate the value of $\ln 1.5$.

Solution 1: We first use Theorem 2.2 to estimate $\ln 1.5$. Since

$$\ln(1+\frac{1}{x})=\ln(x+1)-\ln x,$$

and $\ln(x)$ satisfies the conditions of Theorem 2.2 on $[x,x+1]$, we have

$$\ln(x+1)-\ln(x)=\xi^{-1}\ ,\ \xi\in(x,x+1)\ .$$

Note that $\frac{1}{x+1}<\frac{1}{\xi}<\frac{1}{x}$, hence $\frac{1}{x+1}<\ln(1+\frac{1}{x})<\frac{1}{x}$. Therefore, $\frac{1}{3}<\ln 1.5<\frac{1}{2}$.

Obviously, it is fast to use Theorem 2.2 to estimate the value of ln1.5, but the estimation is not good. Next, we will use Theorem 2.3 to estimate the value of ln1.5.

Solution 2: By Theorem 2.3, $\ln(1+x)$ has the following expansion at $x=0$:

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+\frac{\theta x^{n+1}}{n+1}\ ,$$

Where $0<\theta<1$ and $-1<x\le 1$. Therefore, we can get

$$\ln 1.5\approx\frac{1}{2}-\frac{1}{8}+\frac{1}{24}-\frac{1}{64}=\frac{77}{192}.$$

It can seen that the value of ln1.5 estimated using Theorem 2.3 is much more accurate than the value estimated using Theorem 2.2, but the computational procedure is a bit more complicated. Choosing different theorems for different situations will help us immensely in solving problems.

3.2 Convexity Judgment of Functions

Example 3.2 Let $f(x)$ be a function with second order continuous derivatives on $[c,d]$ and $f''(x)>0$ for each $x\in(c,d)$. Show that $f(x)$ is concave on $[c,d]$.

Solution 1: Let $x_1,x_2\in[c,d]$ with $x_1<x_2$. Denote $x_0=\frac{x_1+x_2}{2}$. Then $x_1=x_0-l$ and $x_2=x_0+l$, where $l=x_0-x_1$. We first introduce the use of Lagrange's mean value theorem to prove the concavity of the function.

(i) By Theorem 2.2 ,

$$f(x_0+l)-f(x_0)=f'(x_0+t_1 l)l\ , \qquad (1)$$

$$f(x_0)-f(x_0-l)=f'(x_0-t_2l)l\,, \qquad (2)$$

where $t_1,t_2\in(0,1)$. Use equation (1)-(2), we have

$$f(x_0+l)+f(x_0-l)-2f(x_0)=[f'(x_0+t_1l)-f'(x_0-t_2l)]l\,.$$

Note that $f'(x)$ is continuous in $[x_0-t_2l, x_0+t_1l]$ and derivable in (x_0-t_2l, x_0+t_1l). Hence by Theorem 2.2,

$$[f'(x_0+t_1l)-f'(x_0-t_2l)]l=f''(\xi)(t_1+t_2)l^2\,,$$

where $\xi\in(x_0-t_2l, x_0+t_1l)$. Recall that $f''(x)>0$ for every $x\in(c,d)$. Therefore

$$f(x_0+l)+f(x_0-l)-2f(x_0)>0\,.$$

That is

$$\frac{f(x_1)+f(x_2)}{2}>f\left(\frac{x_1+x_2}{2}\right).$$

Hence, $f(x)$ is concave on $[c,d]$.

Next, we will introduce using Taylor's theorem to prove the concavity of the function.

Solution 2: Let $x_1,x_2\in[c,d]$ with $x_1<x_2$ and $x_0=\frac{x_1+x_2}{2}$. By Theorem 2.3,

$$f(x_1)=f(x_0)+f'(x_0)(x_1-x_0)+\frac{f''(\xi_1)}{2!}(x_1-x_0)^2\,, \qquad (3)$$

$$f(x_2)=f(x_0)+f'(x_0)(x_2-x_0)+\frac{f''(\xi_2)}{2!}(x_2-x_0)^2\,, \qquad (4)$$

where $\xi_1\in(x_1,x_0)$ and $\xi_2\in(x_0,x_2)$. Use equation $(3)+(4)$, we have

$$f(x_1)+f(x_2)=2f(x_0)+\frac{(x_1-x_2)^2}{8}(f''(\xi_1)+f''(\xi_2))>2f(x_0)$$

as $f''(x)>0$ for every $x\in(c,d)$.

3.3 Function Limit Solving

Example 3.3 Calculate $\lim\limits_{x\to 0}\frac{\sin x-\tan x}{x^3}$.

Solution 1: We first compute this limit by equivalent infinitesimal substitutions. Note that

$$\lim_{x\to 0}\frac{\sin x-\tan x}{x^3}=\lim_{x\to 0}\frac{-\tan x(1-\cos x)}{x^3}\,,$$

and $\tan x\sim x$, $1-\cos x\sim\frac{1}{2}x^2$ as $x\to 0$. Hence,

$$\lim_{x\to 0}\frac{\sin x-\tan x}{x^3}=\lim_{x\to 0}\frac{-\tan x(1-\cos x)}{x^3}=\lim_{x\to 0}\frac{1}{x^3}\times\left(-\frac{1}{2}x^3\right)=-\frac{1}{2}\,.$$

Solution 2: Next we will use Taylor's theorem to solve this limit. By Theorem 2.3,

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots, \quad x \in (-\infty, +\infty);$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots, \quad x \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$

Therefore,

$$\lim_{x\to 0}\frac{\sin x - \tan x}{x^3} = \lim_{x\to 0}\frac{\left(x - \frac{x^3}{3!}\right) - \left(x + \frac{x^3}{3}\right) + o(x^3)}{x^3} = -\frac{1}{2}.$$

As one can see, when using the equivalent infinitesimal substitution, we need to first reduce the fractional equation to the form of multiplying a number of factors, while this step is not necessary when using Taylor's theorem. A good knowledge of Taylor expansions of common functions will be of great help in solving the limits of functions quickly.

4. Conclusion

Lagrange mean value theorem and Taylor's theorem are closely related, and this paper introduces the unified idea of proving these two theorems by Rolle's theorem, as well as the applications of Lagrange mean value theorem and Taylor's theorem in numerical estimation, judgment of concavity and convexity of functions, and solving the limits of functions. It can be seen that the flexible choice of Lagrange mean value theorem and Taylor's theorem for different problems will provide great convenience for problem solving.

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