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Inquiry-based Teaching in Real Variable Function

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Abstract

The course of real variable function is one of the important specialized courses in mathematics and the foundation for studying modern analysis theory. The course boasts a complete theoretical system, and its proof process is characterized by constructiveness and innovativeness. In this paper, we construct an inquiry-based teaching design of measure concept, which can be used as a case of real variable function inquiry-based teaching.

Keywords

Real Variable Function, Inquiry-based Teaching, Measure

1. Introduction

The real variable function is a discipline with abstractness, strong logic, and strict thinking. Students can gradually develop students' thinking logic ability when they study the real function curriculum because of the differences in the way they deal with problems. In order to improve the effect of classroom learning, teachers should adopt such methods as giving the first lesson of real variable function, enriching the interest of classroom teaching and setting up suitable topics for students to effectively improve the efficiency of classroom learning and improve the quality of classroom teaching. Before the course content begins, students should have a broad understanding of the course. Specifically, students should be clear about why they are taking this course. (1) Why is it produced? (2) What is the key to solving the problem? (3) What is the difficulty of the problem? Regardless of the number of teaching hours, this process is essential because it involves an understanding of the overall structure of the course. If the teaching hours are abundant, it is necessary to elaborate, deduce important thinking methods in detail through in-depth analysis of the problem, and obtain related concepts and theorems.

2. Teaching Design

2.1 Guide the Structure of the Real Variable Function Course

What kind of course the real variable function is, what kind of problem it deals with, is the first thing a beginner should know. In 1902, French mathematician Lebesgue published “Integral, Length, Area”, using the measure concept based on set theory to establish Lebesgue integral, thus forming a new branch of mathematics. In teaching, it plays a decisive role to let students grasp the basic ideas and knowledge structure of this course. The integral we learn in mathematical analysis is the Riemann integral, which has obvious limitations, mainly manifested in: (i) The range of function classes that can be integrated in the Riemann sense is too small. (ii) The conditions for the commutative order of Riemann integrals and limits are too strict. (iii) The integral operation is not exactly the inverse of the differential operation. The immediate purpose of the real variable function course is to improve integrals, and Lebesgue abandons partitioning the domain of a given function. Instead, the range of the function is segmented so that each small set divided by the domain consists of points with similar function values. But Lebesgue’s idea of integrals brings a new set of problems. Each small set is not necessarily an interval, it may be a scattered and chaotic set of points and their unions. The first task is to solve the problem of the “measuring tool”, the ”tool” is the Lebesgue measure. The above ideas are realized. Lebesgue measure and Lebesgue integral theory are established. Due to the extensive use of point set analysis method, the necessary point set theory is needed. And dealing with point sets requires the necessary set theory knowledge.

2.2 Focus on the Main Ideas and Methods

The real variable function is a course that links the preceding and the following courses. On the one hand, it is the continuation, development, deepening and extension of the course of mathematical analysis. On the other hand, it is also the basis of functional analysis, partial differential equations, probability theory and stochastic processes. This is the connection between functions of real variables and other courses. Riemann integrals are an important part of mathematical analysis, while Lebesgue integrals are a major part of variable real function theory. These two kinds of integrals have obvious differences in thought and form, but they are also deeply related. Lebesgue integral is a transformation of the essence of Riemann integral, which is carried out on the basis of understanding the defects of Riemann integral. First of all, Riemann integrals have high requirements for the continuity of functions, and the set of discontinuities of Riemann integrable functions can only be the set of zero measures. Second, since the limit function of the Riemann integrable function column is not necessarily Riemann integrable. This makes it necessary to attach some strong conditions to the order of the exchange integral and limit. Lebesgue integral is a revolution of Riemann integral based on the introduction of new concepts such as Lebesgue measure and Lebesgue measurable function. Different from Riemann integral, Lebesgue integral is a new integral sum obtained by dividing the range. Similarly, the concept of continuous functions in mathematical analysis is closely related to measurable functions. Real variable functions are closely related to functional analysis. For example, $L^p[a,b]$ space and conjugate space of $C[a,b]$. The axiom system of modern probability theory is built on the basis of measure theory. There are numerous examples of fractals in functions of real variables.

2.3 Improve Student Participation in Class

Students should participate in discussions in class. A student's question makes everyone think together, and the thought process itself deepens the understanding of the knowledge and provides more ideas for thinking about the problem. For example, explain why the set of rational numbers can be "counted" one by one, which is much denser than the integers, but the "length" is zero, while the set of irrational numbers cannot be counted one by one, and discuss the Cantor triples and the construction of Cantor functions. It should be noted that the gradient of the questions asked in class should not be too high, and it should be gradually deepened. Otherwise, if the problem setting is too difficult, students will have no way to start, lose their interest in active thinking, and only wait for the teacher's explanation, which will make students dependent and completely lose their initiative. Too many easy questions will make students despise the attitude, too many difficult questions will make students tired of learning, so teachers should grasp the degree, carefully select examples and exercises is crucial. Some of the more important content to take exercise classes, further strengthen the understanding and use of knowledge. Exercises class should select some typical examples, use different methods to consider problems from different angles, and actively mobilize students to use various knowledge to solve problems comprehensively.

3. Inquiry-based Teaching

The establishment of measure theory in Euclidean space $\mathbb{R}^n$ needs to solve at least the following problems:

• In $\mathbb{R}^n$, how to measure the size of a set (the establishment of the concept of outer measures)?

• Whether the outer measure is the measure we need (the discovery of unmeasurable sets)?

• What are the criteria for determining measurable sets (the discovery of internal measurements and Caratheodory's Criterion)?

• Whether the measurable set has a length, area, and volume similar to the classical ones (the properties of a measurable set)?

• How large is the extension of the measurable set (the structure of the measurable set)?

If the above problems are solved, measure theory is established, and this is what we have said inquiry teaching. The actual classroom teaching revolves around the following questions.

Question 1: How to measure a general set E in $\mathbb{R}^n$?

In the low-dimensional Euclidean space, the common measurement methods are the length of the interval, the area of the plane area and the volume of the space. However, this approach does not apply to general sets, such as Cantor triples. So how should a general set be measured? As can be seen from the definition of the Riemann integral, it is essentially a number of small rectangles "from the outside" (or “from the inside”) enclosing (approximating) each of the divided small regions, summing and taking off the (upper) certainty. Therefore, we can use some open cuboids to “wrap” a general set E, and then find the lower bound of the open cuboid “volume” sum, that is, the Lebesgue outer measure of set E.

Definition Let $E \subset \mathbb{R}^1$. An outer measure μ^* of E is defined by

$$\mu^*(E) = \inf\left\{\sum_{j=1}^{\infty} |I_j| : \text{ All } \{I_j\} \text{ satisfying } A \subset \cup_j I_j\right\},$$

where $I_j = [a_j, b_j], j \geq 1$ are closed intervals and $|I_j| = b_j - a_j$.

Is this a reasonable measure? There are at least two dimensions to consider: (i) Does the new measure cover the old measure? In other words, for intervals, rectangles, or cuboids, are the old and new metrics the same? (ii) Are properties of length, area, or volume inherited? What are the common characteristics of length, area and volume? It is not difficult to abstract out the common characteristics of these three concepts: non-negative, monotonicity and additivity. Do Lebesgue out measures also have these three properties?

According to Lebesgue’s definition of outer measures, non-negativity and monotonicity are almost obvious, and the “countable subadditivity” of out measures is not difficult to obtain, i.e.,

(1) $\mu^*(E) \geq 0$;

(2)Monotonicity: $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$;

(3)Countable subadditivity: $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$.

Because of the existence of unmeasurable sets, there are always some sets that are not countably additive. The only thing to do is to exclude these sets and consider only those that are additive. The question is, how do you tell if a set is measurable (additive)? This naturally raises the following question.

Question 2: What sets have measure countably additivity? How to determine?

Here, students can still be guided to think of the Riemann integral. A series of small rectangles are used to approximate the curved trapezoid from the inside and outside, and the small sum and large sum of the curved trapezoid are obtained respectively. Intuitively, the Lebesgue out measure is similar to the Darboux grand sum, a measure that approximates the set E from the outside. Thus, it may be considered here to define an “inner measure” corresponding to the small sum of Darboux, so that when the inner and outer measures are equal, the set is called a measurable set. This is a natural move from the idea of Riemann integrals, and it is suitable for one-dimensional Spaces because the open set on a line has a structural theorem, which is technically easy. However, in the n-dimensional Euclidean space $\mathbb{R}^n$, it is more troublesome, because the open set in the higher-dimensional space is far less straightforward than in the one-dimensional case, and in fact, it cannot be represented as the union of disjoint open cuboids, which determines the complexity of the closed set structure theorem in the higher-dimensional case.

From a logical point of view, the introduction of this definition is natural, but it is difficult to see the inherent structure and properties of measurable sets from the Castillo condition. The following basic questions need to be answered.

• Are common sets measurable?

• Are measurable sets compatible with the usual concepts of length, area and volume?

• Are measurable sets closed to operations on sets?

• Does it really satisfy additivity?

The answers to the above questions are almost trivial, except that the question of whether cuboid volume and measure are equal is a bit cumbersome, but they do help students have a preliminary experience of measurable sets and understand that the concept is in line with human intuition. However, before answering the question whether the open set and the closed set can be measured, it is necessary to answer the last two of the above four questions first to measure the nature of the set, so naturally transferred to the study of the following questions.

Question 3: Are measurable sets closed to operations on sets? What about limit operations on sets?

The difficulty of this problem is how to build a bridge between the two sets E_1 and E_2 satisfying the severability condition respectively and the union (or intersection or difference) satisfying the severability condition of the two sets. Take the union of two measurable sets as an example, although the problem itself has a certain abstractness, it can still analyze the relationship between them by means of geometric intuition through Wayne diagram.

After the case of two sets is proved, the closure of measurable sets with respect to the operation of union, intersection, difference and complement of sets can be easily extended to the case of finite number, i.e.,

Corollary Let $E \subset \mathbb{R}^1$. If $\{E_n\}_{n=1}^{\infty}$ is a countable disjoint collection of sets, then

$$\mu^*\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \mu^*(E_n).$$

According to the definition of limit of set sequence and the closure of measurable set to set operation, it is easy to see that the limit of measurable set sequence is still measurable.

Question 4: Can the limits and measures of a measurable set sequence swap orders?

As can be seen from the limit definition of the sequence of sets, the general case is not difficult if the problem can be solved for the monotonic series of sets.

• If $\{E_n\}_{n=1}^{\infty}$ is an increasing sequence of measurable sets, does equation

$$\mu^*(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} \mu^*(E_n)$$

hold?

• If $\{E_n\}_{n=1}^{\infty}$ is a decreasing sequence of measurable sets, does equation

$$\mu^*(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} \mu^*(E_n)$$

hold?

A useful implication of the countable additivity of a measure is the following monotonicity result.

Theorem Let $E \subset \mathbb{R}^1$. If $\{A_n\}_{n=1}^{\infty}$ is an increasing sequence of measurable sets, meaning that $A_n \subset A_{n+1}$, then

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu^*(A_n).$$

If $\{A_n\}_{n=1}^{\infty}$ is s a decreasing sequence of measurable sets, meaning that $A_{n+1} \subset A_n$, then

$$\mu^*(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu^*(A_n).$$

Proof. If $\{A_n\}_{n=1}^{\infty}$ is an increasing sequence of sets and $B_n = A_{n+1} \setminus A_n$, then $\{B_n\}_{n=1}^{\infty}$ is a disjoint sequence with the same union, so by the countable of μ^*

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) = \mu^*(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu^*(B_n).$$

Moreover, since $A_j = \bigcup_{i=1}^{j} B_i$, $A_j = \bigcup_{i=1}^{j} B_i$,

$$\mu^*(A_j) = \sum_{i=1}^{j} \mu^*(B_i),$$

which implies that

$$\sum_{i=1}^{\infty} \mu^*(B_i) = \lim_{j\to\infty} \mu^*(A_j)$$

and the first result follows. The second proof is similar and is left to the student.

After solving the above four problems, Lebesgue measure theory is basically established.

4. Conclusion

Inquiry-based teaching methods can be effective in helping students understand the concept of outer measures in real variable functions. By providing students with opportunities to actively explore, collaborate, and think critically about the concept, instructors can facilitate deeper learning and enhance student outcomes. Future research could explore additional strategies for implementing inquiry-based teaching in the context of real variable functions and measure theory.

This paper highlights the potential of inquiry-based teaching to transform the learning experience in advanced mathematical concepts such as outer measures in real variable functions. By embracing a student-centered approach, educators can empower students to take ownership of their learning and develop the skills necessary to succeed in higher mathematics.

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