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Application of Norm Theory in Linear Algebra

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Abstract
By using the basic properties of matrix norm and the equivalence of different norms, we analyze the singular value decomposition and judgment problem of invertible matrices by examples, which provide new ideas for solving linear algebra problems.
Keywords
Matrix norm, Singular value decomposition, Matrix inversion, Linear algebra

1. Introduction

The norm theory is an important tool for research in numerical computation, algorithm optimization, error analysis and so on (Chen Feixiang & Wu Zhongxiang, 2009; Zou Limin, 2016; Shen Jinzhong & Deng Liubao, 2016). And vectors and matrices are important concepts throughout the knowledge points of linear algebra courses, so the study of some theories in linear algebra with the help of paradigm theory has become a topic of concern for scholars. Ren Fangguo et al. (2012) inscribed random matrices from the perspective of paradigms and derived inequalities for random matrix paradigms and the sufficient conditions for their bounds, laying the foundation for the study of random matrix tensor product paradigms, while Shi Nisha (2013) obtained the minimal paradigm solution of a compatible system of linear equations with the help of vector paradigms. Dongxia Tian (2020) also studied the compatibility of vector paradigms with matrix paradigms. In this paper, we defined matrix paradigms from different perspectives and proved two important problems of singular value decomposition and matrix invertibility in linear algebra by using matrix paradigms, which opens up a new way of thinking for the teaching and learning of linear algebra.

2. Background Knowledge

Definition 1: Let a matrix $A \in R^{m\times n}$, defining a real-valued function $\|A\|$ satisfy the following four conditions:

(1) Non-negativity: $\|A\| \geq 0$, If and only if $A=0$, then $\|A\|=0$ 。

（2）Homogeneity: For any $\lambda \in R$, there is $\|\lambda A\| = |\lambda|\|A\|$。

（3）Triangle inequality: triangle inequality: For any $A, B \in R^{m\times n}$, there is $\|A+B\| \le \|A\| + \|B\|$。

（4）Compatibility: For any $A, B \in R^{m\times n}$, there is $\|AB\| \le \|A\|\|B\|$。

Then $\|A\|$ is said to be a matrix paradigm for b A.

Since $R^{m\times n}$ and R^{mn} are homomorphic and a $m\times n$ real matrix can be viewed as a mn -dimensional real vector, the Frobenius paradigm for matrices is commonly used.

Definition 2: Frobenius paradigm(F —paradigm) $\|A\|_F = (\sum_{i=1}^{m}\sum_{i=1}^{n}\left|a_{ij}\right|^2)^{\frac{1}{2}}$

Obviously, the F-norm is invariant under orthogonal transformations. That is, for an $m\times n$ -order matrix A, if there exists an m-order orthogonal matrix Q and an n-order orthogonal matrix P, then $\|QAP\|_F = \|A\|_F$.

In addition, there is also a useful norm, $H\ddot{o}lder$ paradigm.

Definition 3: If is an n-dimensional column vector $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T$ and $\|\boldsymbol{x}\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$, then $\|A\|_p = \sup_{\boldsymbol{x}\ne\boldsymbol{0}} \frac{\|A\boldsymbol{x}\|_p}{\|\boldsymbol{x}\|_p}$ is called the $H\ddot{o}lder$ paradigm of the matrix A.

Similarly, there is also $\|QAP\|_2 = \|A\|_2$ (The definitions of Q and P are the same as above).

Theorem 1 (Gan Wenzhen, Chen Jianhua, & Zhu Peng., 2012) Let $\|\cdot\|$ be a matrix norm on $C^{n\times n}$. If $A = (a_{ij})_{n\times n} \in C^{n\times n}$, then any two matrix norms on $C^{n\times n}$ are equivalent.

3. The Applications of Vector Norms in the Singular Value Decomposition Theory

The theory of matrix norms and the theory of vector norms can be used to prove one of the most important decompositions in matrices—the singular value decomposition.

Definition 4: Let A and B be real matrices of order $m\times n$. If there exist real orthogonal square matrices Q_1 of order m and Q_2 of order n such that, then matrices A and B are said to be orthogonally equivalent.

Definition 5: Let A be a real matrix of order $m\times n$. Then the arithmetic square root of the non-zero eigenvalues of the square matrix $A^{\mathrm{T}}A$ of order n is called the singular value of the matrix A.

Theorem 2 (Li Tongsheng, Zha Jianguo, & Wang Xinmao, 2010) Let $\mu_1, \mu_2, \cdots, \mu_r$ be all the singular values of the matrix A of order $m \times n$, where $u_1 \geq u_2 \geq \cdots u_r > 0$. Let the rank A of be $r = rank(A),$. There exist matrices $U = (\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m) \in R^{m\times m}$ and $V = (\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n) \in R^{n\times n}$ such that $U^{\mathrm{T}}AV{=}diag((diag(\mu_1, \mu_2, \cdots, \mu_r), \mathbf{0})$, where and $\mathbf{0}$ is the zero matrix of order $(m-r)\times(n-r)$.

Proof: Let $\boldsymbol{x} \in R^n, \boldsymbol{y} \in R^m$ satisfy $\|\boldsymbol{x}\|_2 = \|\boldsymbol{y}\|_2 = 1$ and $\mu = \|A\|_2$. Let $V = (\boldsymbol{x}, V_1) \in R^{n\times n}$ and $U{=}(\boldsymbol{y}, U_1) \in R^{m\times m}$ be orthogonal (because an orthonormal basis of a vector can always be extended to an orthonormal basis of the entire space). Then $U^T AV$ has the following structure:

$$A_1{=}U^T AV = \begin{pmatrix} \mu & \omega^T \\ 0 & B \end{pmatrix}.$$

Since $\left\| A_1 \begin{pmatrix} \mu \\ \boldsymbol{\omega}^T \end{pmatrix} \right\|_2^2 \geq (\mu^2 + \omega\omega^T)^2$,then $\|A_1\|_2 \geq \mu^2 + \omega\omega^T$.

Since $\mu^2 {=} \|A\|_2^2$ and $\|A_1\|_2 = \|A\|_2$, then $\omega^T\omega{=}0$, and thus $\omega{=}0$.

It can be proved that $U^{\mathrm{T}}AV{=}diag((diag(\mu_1, \mu_2, \cdots, \mu_r), \mathbf{0})$ by using mathematical induction.

In document (Gan Wenzhen, Chen Jianhua, & Zhu Peng., 2012), Theorem 2 was proved by decomposing the semi-positive definite symmetric square matrix $A^T A$ of order n, and then utilizing the properties of block matrices and orthogonal matrices. However, the proof process is rather skill-intensive and generally not easy for beginners to grasp. In contrast, the proof method based on vector norms mentioned above is simpler and easier to understand.

4. Applications of Vector Norms in Judging the Invertibility of Matrices

The proof and application of inverse matrices are important knowledge points in linear algebra. Generally, the invertibility of a matrix is judged by means of the non-zero determinant of a square matrix, elementary transformations, rank and other methods. In fact, the invertibility of a matrix can also be judged from the perspective of vector norms.

Example 1: Let $A \in R^{n\times n}$ be a matrix, and for a certain matrix norm on $R^{n\times n}$, if $\|A\| < 1$, then the matrix $I - A$ is invertible.

Proof: For any $\boldsymbol{x} \neq \boldsymbol{0}$, since $\|A\| < 1$, we have $\|A\boldsymbol{x}\| \le \|A\|\|\boldsymbol{x}\| < \|\boldsymbol{x}\|$.

So $\|(I - A)\boldsymbol{x}\| \ge \|\boldsymbol{x}\| - \|A\|\|\boldsymbol{x}\| > 0$. Due to the non-negativity of matrix norms, we know that $(I - A)\boldsymbol{x} \neq \boldsymbol{0}$. Therefore, the matrix $I - A$ is invertible.

Example 2: Let $A \in R^{n\times n}$ be an invertible matrix and $B \in R^{n\times n}$ be a matrix. If for a certain matrix norm on $R^{n\times n}$, we have $\|A^{-1}B\| < 1$, then the matrix $A + B$ is invertible.

Proof: It follows from Example 1 that the matrix $I + A^{-1}B$ is invertible. Consequently, the matrix $I + A^{-1}B$ is invertible.

The above two examples determine the invertibility of matrices from the value range of norms. On this basis, we can further discuss the convergence and divergence of matrix sequences, and further research will be carried out in the future.

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