

## *Original Paper*

# Penetration Strategies of the Number-Shape Combination

## Thought in Middle School Mathematics Teaching

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### **Abstract**

*The number-shape combination is one of the most fundamental and important methods of thought in middle school mathematics. It organically combines quantitative relationships with spatial forms, serving as both a core method of mathematical research and a key competency in mathematical learning. This paper systematically explores the penetration strategies of the number-shape combination thought in middle school mathematics teaching. First, it expounds the connotation and educational value of the number-shape combination thought, pointing out its unique role in reducing cognitive difficulty, cultivating thinking ability, and promoting knowledge transfer. Second, it analyzes the prominent problems existing in the penetration of the number-shape combination thought in current middle school mathematics teaching, including teachers' cognitive deviations, single teaching methods, and students' difficulties in transformation. On this basis, five penetration strategies are proposed: the two-way construction of number-shape combination in concept teaching, the flexible application of number-shape combination in problem-solving teaching, the characteristic application of number-shape combination in different modules, the empowerment of number-shape combination teaching by information technology, and the systematic training and evaluation of the number-shape combination thought. This paper aims to provide theoretical guidance and practical reference for frontline middle school mathematics teachers to effectively penetrate the number-shape combination thought, promoting the comprehensive improvement of students' mathematical core competencies.*

### **Keywords**

*Number-Shape Combination, Middle School Mathematics, Teaching Strategies, Mathematical Method of Thought, Core Competencies*

## 1. Introduction

### 1.1 Research Background

#### 1.1.1 The Importance of Teaching Mathematical Methods of Thought

With the in-depth advancement of basic education curriculum reform, the goal of mathematics education has shifted from simple knowledge transmission to the cultivation of students' mathematical core competencies. Both the *Compulsory Education Mathematics Curriculum Standards (2022 Edition)* and the *General High School Mathematics Curriculum Standards (2017 Edition, 2020 Revision)* clearly state that mathematics education should value the penetration of mathematical methods of thought, enabling students to understand and master the basic thoughts of mathematics and form a systematic mathematical thinking framework. Mathematical methods of thought are the essence and soul of mathematical knowledge and the bridge connecting basic knowledge and higher-order abilities. Among the many mathematical methods of thought, the number-shape combination holds a very important position due to its foundational, universal, and practical characteristics.

#### 1.1.2 The Difficulties Faced by Middle School Mathematics Teaching

There are several common problems in current middle school mathematics teaching: students have difficulty understanding abstract mathematical concepts, often knowing "what it is" but not "why it is"; they mechanically apply formulas when solving problems, lacking flexible adaptability; different mathematics modules (algebra, geometry) are fragmented, making it difficult for students to establish a unified cognitive system. A deep-seated reason for these problems is the lack of the number-shape combination thought. Students are accustomed to "pure algebra" or "pure geometry" one-way thinking, are not good at flexible conversion and translation between numbers and shapes, leading to cognitive limitations and thinking set patterns.

#### 1.1.3 The Unique Value of the Number-Shape Combination Thought

The value of the number-shape combination thought lies in its organic integration of abstract quantitative relationships and intuitive spatial forms, leveraging the dual advantages of the "precision" of numbers and the "intuitiveness" of shapes. The famous Chinese mathematician Hua Luogeng once incisively stated: "Numbers lack shapes, losing intuitiveness; shapes lack numbers, failing to capture subtleties; number-shape combination is best in every way; separating the two, everything comes to a halt." This statement profoundly reveals the significance of number-shape combination for mathematical learning and research. Penetrating the number-shape combination thought in teaching helps students turn abstraction into concreteness, complexity into simplicity, and obscurity into clarity, serving as an effective way to break through teaching difficulties.

### 1.2 Research Purpose and Significance

#### 1.2.1 Research Purpose

This study aims to systematically construct a system of strategies for penetrating the number-shape combination thought in middle school mathematics teaching. Specific goals include: First, to clarify the connotation of the number-shape combination thought and its value positioning in middle school

mathematics teaching; Second, to diagnose the main problems in current number-shape combination teaching and their causes; Third, to propose systematic and operable penetration strategies based on different teaching links and content modules; Fourth, to provide ideas for the teaching evaluation of the number-shape combination thought.

### 1.2.2 Theoretical Significance

This study helps enrich and develop the theoretical system of teaching mathematical methods of thought. As a discourse in mathematics education with Chinese characteristics, the number-shape combination has long received widespread attention but lacks systematic strategy research. This paper attempts to construct a well-structured and highly operable strategic framework, which can serve as a theoretical reference for subsequent research.

### 1.2.3 Practical Significance

From a practical perspective, this study can provide specific paths and methodological guidance for frontline middle school mathematics teachers to penetrate the number-shape combination thought in classroom teaching. The strategies are refined based on teaching practice, taking into account the characteristics of different school stages (junior high, senior high) and different modules (algebra, geometry, functions, analytic geometry), possessing strong practicality and transferability.

## 1.3 Domestic and International Research Status

### 1.3.1 Domestic Research Status

Domestic research on the number-shape combination thought can be traced back to the 1980s, but systematic pedagogical research has mainly concentrated in the last two decades. Existing research primarily covers the following aspects: first, the philosophical basis and cognitive psychology explanation of the number-shape combination thought; second, case studies of number-shape combination applications in specific knowledge points (such as functions, equations, inequalities); third, exploration of teaching strategies for cultivating number-shape combination ability. Overall, case studies are abundant but theoretical enhancement is insufficient; strategic suggestions are numerous but lack an operable framework.

### 1.3.2 Relevant International Research

There is no completely corresponding concept of "number-shape combination" in international mathematics education research, but related research involves themes such as "multiple representations," "visualization," and "geometric intuition." Duval's theory of representation conversion emphasizes that mathematical understanding depends on the ability to convert between different representation systems for the same mathematical object, which highly aligns with the idea of number-shape combination. Arcavi's research on the relationship between symbolization and visualization also provides theoretical support for number-shape combination teaching.

### 1.3.3 Limitations of Existing Research

Overall, existing research has the following shortcomings: First, the emphasis on the two-way conversion between "number" and "shape" is unbalanced, with most studies focusing on "using shapes to aid

numbers" and insufficient attention to "using numbers to aid shapes"; Second, strategy research is rather fragmented, lacking systematic construction from the whole process of design, implementation, and evaluation; Third, there is a lack of operational frameworks for evaluating the level of students' number-shape combination ability.

#### *1.4 Research Methods and Content Framework*

##### 1.4.1 Research Methods

This study primarily uses the literature research method and case analysis method. Through systematic review of relevant literature on the number-shape combination thought, core concepts and theoretical foundations are refined; through collecting and analyzing typical teaching cases, effective penetration strategies and operational methods are summarized.

##### 1.4.2 Content Framework

This paper is divided into five parts. Part 1 is the introduction, clarifying the research background, purpose, significance, and current state. Part 2 discusses the connotation and value of the number-shape combination thought. Part 3 analyzes the problems existing in current teaching. Part 4 proposes systematic penetration strategies (five key points). Part 5 presents the conclusions and recommendations.

## **2. The Connotation and Educational Value of the Number-Shape Combination Thought**

### *2.1 The Essential Connotation of the Number-Shape Combination Thought*

#### 2.1.1 The Dialectical Relationship between "Number" and "Shape"

The objects of mathematical research can be summarized into two major categories: "number" and "shape." Number is abstract, logical, and precise; shape is intuitive, visual, and holistic. There is a profound internal connection between number and shape: any quantitative relationship can be reflected in some spatial form, and any spatial form contains quantitative relationships. The essence of number-shape combination is to fully utilize the complementarity of these two types of mathematical objects, achieving the unity of "abstraction and intuition," "logic and intuition," and "part and whole."

From a philosophical perspective, number-shape combination embodies the basic principle of dialectical materialist epistemology: from perceptual knowledge to rational knowledge, and then from rational knowledge back to practice. Shape corresponds to the stage of perceptual knowledge, providing intuitive material; number corresponds to the stage of rational knowledge, providing precise depiction; their combination is the complete process of cognition.

#### 2.1.2 The Two Basic Forms of Number-Shape Combination

Number-shape combination includes two basic forms: "using shapes to aid numbers" and "using numbers to aid shapes."

"Using shapes to aid numbers" refers to using the intuitiveness of geometric figures to understand and solve algebraic problems. For example, using the graph of a quadratic function to understand the distribution of roots of a quadratic equation; using the number line to understand the geometric meaning of absolute value; using the unit circle to understand the periodicity of trigonometric functions. When

algebraic problems are too abstract, graphics can provide intuitive support, reducing cognitive load.

"Using numbers to aid shapes" refers to using algebraic methods to solve geometric problems. For example, establishing a coordinate system to represent curves with algebraic equations (analytic geometry), using vector methods to prove geometric theorems, using algebraic calculations to solve for extreme values of geometric quantities. When geometric problems are too complex, algebraic methods can provide procedural operational paths, enhancing the certainty and operability of problem-solving.

The two forms complement each other and bring out the best in each other. Complete number-shape combination ability requires students to be able to convert freely in both directions.

### 2.1.3 The Relationship between Number-Shape Combination and Multiple Representations

Number-shape combination is essentially a special form of multiple representations of mathematical objects. A mathematical object (such as a function) can have multiple representation methods: symbolic representation (analytic expression), graphical representation (image), tabular representation (data table), linguistic representation (text description), etc. Number-shape combination particularly emphasizes the conversion and connection between symbolic and graphical representations. The ability to flexibly switch between different representations is a hallmark characteristic of mathematical understanding.

## 2.2 The Educational Value of the Number-Shape Combination Thought

### 2.2.1 Reducing Cognitive Difficulty and Promoting Concept Understanding

There are many abstract concepts in middle school mathematics, such as opposite numbers, absolute value, functions, extreme values, derivatives, etc. If solely relying on algebraic definitions, students often find it difficult to build intuitive understanding. With the help of geometric figures, these abstract concepts can gain concrete "anchors." For example, the geometric meaning of absolute value is the distance from a point on the number line to the origin. This explanation transforms the abstract "non-negativity" and "piecewise definition" into an intuitive spatial relationship, making it easier for students to accept and remember. Research shows that students receiving number-shape combination teaching outperform those receiving traditional teaching in both the depth of understanding of abstract concepts and retention time.

### 2.2.2 Cultivating Thinking Ability and Enhancing Problem-Solving Skills

Number-shape combination thinking is a higher-order thinking ability that requires students to flexibly convert and coordinate between two symbol systems. This training helps cultivate students' observation ability (discovering quantitative laws from figures), association ability (establishing connections between numbers and shapes), transformation ability (transforming a problem from one form to another), and creativity (choosing the optimal representation method). In problem-solving, number-shape combination can often open up new ideas and find "ingenious solutions."

### 2.2.3 Embodying the Essence of Mathematics and Forming a Disciplinary Perspective

Mathematics is not a simple pile of knowledge from various branches but an organically connected whole. Number-shape combination is an important manifestation of this internal connection. Through repeated penetration of the number-shape combination thought in teaching, students gradually establish the

disciplinary perspective that "mathematics is unified," recognizing the interdependence between algebra and geometry, analysis and synthesis, precision and intuition. This macroscopic disciplinary perspective helps students form a systematic mathematical cognitive structure, promoting long-term retention and transferable application of knowledge.

### **3. Problems in Current Number-Shape Combination Teaching**

#### *3.1 Problems at the Teacher Level*

##### **3.1.1 Cognitive Deviation Regarding the Number-Shape Combination Thought**

Some teachers' understanding of the number-shape combination thought remains superficial, equating it with "drawing pictures to solve problems," failing to recognize its methodological significance as a fundamental mathematical thought. This cognitive deviation leads to the penetration of number-shape combination in teaching being sporadic and accidental, rather than systematic and conscious. Some teachers even reject number-shape combination in teaching, believing that "geometric intuition lacks rigor," insisting on using pure algebraic methods to solve problems, thus missing opportunities to cultivate students' thinking flexibility.

##### **3.1.2 Single and Rigid Penetration Strategies**

Even among teachers who value number-shape combination, the strategies used are often singular. The most common approach is: the teacher demonstrates by drawing a picture during explanation, and students imitate through practice. This "teaching-imitation" model fails to enable students to truly internalize the number-shape combination thought. Students only "see" the result of number-shape combination under the teacher's guidance, rather than experiencing the thought processes of "why think of using number-shape combination," "how to choose the representation for the shape," and "how to convert between numbers and shapes." When faced with new, unfamiliar problems, students still do not know how to apply it.

##### **3.1.3 Insufficient Guidance and Evaluation of Students**

Teachers tend to focus on whether students have "drawn" the figure, neglecting the diagnosis and guidance of students' thought processes in number-shape combination. When a student draws a figure but applies it incorrectly, the teacher directly corrects the result, rarely asking follow-up questions like "What quantitative relationships does your drawing reflect?", "Which features of the figure are key?", or "How do the figure and the algebraic expression correspond?". Regarding evaluation, whether in daily tests or major exams, the assessment of number-shape combination ability is mostly implicitly embedded, lacking dedicated evaluative dimensions, leading to a disconnect between teaching, learning, and assessment.

#### *3.2 Problems at the Student Level*

##### **3.2.1 Weak Graphic Representation Ability**

A considerable number of students lack the ability and habit of drawing accurately. Their drawn figures have disproportionate scales, missing key points, or omitted special cases, causing the failure of "using

shapes to aid numbers." For example, when drawing graphs of quadratic functions, students often neglect key information such as the direction of opening, axis of symmetry, and vertex coordinates, resulting in a graph that cannot reflect the function's properties. Some students even fear drawing, considering it a "waste of time," preferring to rely solely on calculation rather than draw.

### 3.2.2 Difficulty Extracting Quantitative Relationships

"Using numbers to aid shapes" requires students to extract quantitative relationships from geometric figures. However, many students, when facing a figure, do not know which quantities are key or how to express these quantitative relationships using algebraic language. For example, in geometry proof problems, students often "see" the characteristics of the figure (such as perpendicularity, equality) but do not know how to use algebraic tools (such as coordinate systems, vectors) to convert them into computable forms. This difficulty reflects a disconnect between geometric intuition and algebraic manipulation.

### 3.2.3 Lack of Conversion Awareness and Ability

The core of number-shape combination is "conversion." However, most students are accustomed to "locking in" one representation method and lack the awareness to actively convert. When encountering difficulty with one method, they will not actively try the other. This is both a manifestation of thinking set patterns and related to the orientation of teaching towards a "single correct solution." Students develop a mistaken belief: each problem has one "correct method," and finding this method is the entirety of problem-solving. They do not realize that changing perspective is itself an important problem-solving strategy.

## 3.3 Problems at the Teaching Level

### 3.3.1 Segmentation of Teaching Content

In current textbooks and teaching practices, algebraic content and geometric content are often arranged and taught relatively independently. Functions, equations, and inequalities belong to the field of "algebra," while plane geometry and solid geometry belong to the field of "figures and geometry." Although textbooks intentionally set cross-cutting content (such as analytic geometry), the overall pattern remains one of segmented teaching. This segmentation objectively reinforces students' perception of "numbers and shapes being separate," which is not conducive to the overall penetration of the number-shape combination thought.

### 3.3.2 Inappropriate Timing of Penetration

The penetration of number-shape combination requires appropriate timing. Introducing figures too early may interfere with students' accurate understanding of algebraic symbols (e.g., using graphs instead of the definition of a function); introducing them too late may cause students to "take a detour" in understanding abstract concepts. The best timing for penetration for different knowledge points needs careful consideration by teachers. In practice, some teachers either rigidly follow the textbook (using graphics only when the textbook mentions them) or use them randomly (drawing wherever they think of it), lacking careful consideration of the timing.

### 3.3.3 Lack of Systematic Instructional Design

The number-shape combination thought cannot be fully covered in one lesson or one unit; it needs continuous, spiraling penetration throughout the three years of middle school mathematics teaching. However, current teaching lacks "systematic instructional design for the number-shape combination thought": What are the key points for penetration in different grades? What functions do different modules serve? How to evaluate the developmental level of students' number-shape combination ability? These questions lack overall planning and design, resulting in less effective penetration.

## 4. Penetration Strategies of the Number-Shape Combination Thought in Middle School Mathematics Teaching

### 4.1 Two-way Construction of Number-Shape Combination in Concept Teaching

#### 4.1.1 Introducing New Concepts through "Shape"

When introducing abstract mathematical concepts, students' intuitive experience should be fully utilized, starting from "shape." For example, when explaining "opposite numbers," a pair of opposite numbers can be represented on the number line for students to observe their symmetry about the origin; when explaining "function monotonicity," first let students observe the rising and falling trends of the function graph before giving the strict algebraic definition. This sequence of "intuition first, abstraction follows" conforms to students' cognitive laws and can reduce the difficulty of concept understanding.

Key operational points are: the graphic chosen should be typical (avoiding misleading special cases), the correspondence between the graphic and the definition should be clear (pointing out "which part of the graphic corresponds to which definition"), and the transition from graphic to definition should be natural (guiding students to abstract the definition themselves through questioning).

#### 4.1.2 Two-way Conversion Training for Concept Representation

After concepts are established, specialized practice in representation conversion should be conducted. Given the definition of a concept, require students to draw the corresponding figure; or given a figure, require students to write the corresponding algebraic description. For example, given that the solution set of the quadratic inequality  $ax^2 + bx + c > 0$  is  $x < x_1$  or  $x > x_2$ , require students to draw an approximate graph of the corresponding quadratic function and determine the sign of  $a$ . This type of two-way conversion training is the "basic skill" for strengthening number-shape combination awareness. Practice should progress from simple to complex, from concrete to general, gradually increasing the complexity and abstraction of the conversion. Initially, specific numerical values can be given, later transitioning to expressions containing parameters.

#### 4.1.3 Connecting Number-Shape Combination in the Concept Network

After individual concepts are established, help students construct concept networks, revealing the connections between different concepts through number-shape combination. For example, connect the three concepts of "equation," "inequality," and "function": The solution of the equation  $f(x) = 0$  is the x-coordinate of the intersection point of the graph of function  $y = f(x)$  with the x-axis; the solution set

of the inequality  $f(x) > 0$  is the range of  $x$  corresponding to the part of the function graph above the  $x$ -axis. This type of connection not only deepens the understanding of each concept but also reveals the internal relationships between them.

#### *4.2 Flexible Application of Number-Shape Combination in Problem-Solving Teaching*

##### 4.2.1 "Graphicalization" in the Problem Analysis Stage

When analyzing mathematical problems, students should develop the habit of "thinking of graphics when seeing a problem." Specific practices include: during the problem-understanding stage, re-express the problem information using graphics; during the condition analysis stage, use graphics to represent the relationships between given conditions and unknown quantities; during the idea exploration stage, use graphics to seek breakthroughs for problem-solving. This graphicalization can be done on scratch paper, not requiring precise geometric construction using tools, but should capture the essential structure of the problem.

In example problem explanations, teachers should "think aloud," demonstrating the thought process of how they came up with using graphics as an aid, rather than just showing the result of drawing the graphic.

##### 4.2.2 Comparison and Optimization of Multiple Problem-Solving Paths

For problems solvable by multiple methods, consciously compare the advantages and disadvantages of algebraic methods and geometric (graphical) methods. For example, to find the minimum value of the function  $y = \sqrt{x^2 + 1} + \sqrt{(x - 3)^2 + 1}$ , algebraic methods involve complex radical operations, while the geometric method transforms it into the sum of distances from a moving point on the  $x$ -axis to two fixed points, obtaining the answer immediately using the "general's horse drinking" model. Through comparison, students not only master specific problem-solving techniques but also appreciate the value of number-shape combination in optimizing problem-solving approaches.

When comparing, one should discuss: When is the algebraic method more efficient? When is the geometric method simpler? Can the two methods be combined? These discussions help cultivate students' metacognitive abilities.

##### 4.2.3 Enhancing Number-Shape Combination through Problem-Solving Reflection

Reflection after solving problems is an important link for enhancing number-shape combination ability. Teachers should guide students to reflect: Was number-shape combination utilized in solving the problem? What key information did the graphic provide? Was there a better graphic representation? Could the problem be solved without using graphics? Did using graphics deepen understanding of the problem? This reflection elevates individual problem-solving experiences into general methodological understanding.

Establishing a habit of "one problem, one reflection" can include a dedicated "Reflection on the Application of Number-Shape Combination" segment in homework or exam review sessions.

#### *4.3 Characteristic Application of Number-Shape Combination in Different Modules*

##### 4.3.1 Number-Shape Combination in Functions and Equations

Functions are the field where the number-shape combination thought is most fully manifested in middle

school mathematics. Every function has a corresponding graph, and the properties of functions (domain, range, monotonicity, parity, periodicity, extreme values) are intuitively reflected in the graph. In teaching, one should achieve "inseparable association between analytic expression and graph": when learning a new function, immediately study its graph; when given a graph, immediately describe it using algebraic language. Problems combining equations and functions, such as zero point problems and root distribution problems, are typical applications of "using shapes to aid numbers."

Specialized training can be designed as a series of exercises on "No graph, no function," requiring that a sketch be drawn for every function problem to aid analysis.

#### 4.3.2 Number-Shape Combination in Inequalities

The solution sets of inequalities can be intuitively represented on a number line or in a Cartesian coordinate system. The solution set of a linear inequality in one variable appears as a ray or interval on the number line; the solution set of a quadratic inequality corresponds to the "above/below" relationship of the quadratic function graph; linear programming problems represent the solution set of a system of inequalities as a planar region. In teaching, the habit of "draw the number line whenever solving an inequality" should be cultivated to make abstract inequality relationships intuitive.

For inequality problems with parameters, graphics can help with case classification. Dynamically demonstrating the impact of parameter changes on the boundaries of the solution set is an important method for breaking through difficult points.

#### 4.3.3 Number-Shape Combination in Plane Geometry and Analytic Geometry

Analytic geometry is a direct product of the number-shape combination thought. By establishing a coordinate system, geometric figures obtain algebraic equations, and geometric problems are transformed into algebraic problems. In teaching, the unity and procedural nature of the "coordinate method" should be emphasized, helping students establish the operational process of "figure  $\rightarrow$  coordinates  $\rightarrow$  equation  $\rightarrow$  calculation  $\rightarrow$  conclusion."

At the same time, attention should also be paid to the application of "using shapes to aid numbers" in geometry. For complex algebraic expressions, intuitive understanding can be achieved by endowing them with geometric meaning. For example,  $\sqrt{(x-a)^2 + (y-b)^2}$  can be understood as the distance from point  $(x, y)$  to the fixed point  $(a, b)$ , an insight very useful when solving extreme value problems.

### 4.4 Empowering Number-Shape Combination Teaching with Information Technology

#### 4.4.1 Intuitive Demonstration with Dynamic Geometry Software

Dynamic geometry software such as GeoGebra and The Geometer's Sketchpad provides powerful tools for number-shape combination teaching. Teachers can use these software tools to dynamically demonstrate: the continuous change of a function graph as parameters change, the trajectory of a moving point, the transformation process of figures, etc. This dynamic demonstration makes "static" figures "move," helping students understand the laws of change embedded in figures and establishing the dialectical perspective of "movement within stillness, stillness within movement."

For example, when explaining the influence of  $a$ ,  $b$ , and  $c$  on the graph of the quadratic function  $y =$

$ax^2 + bx + c$ , using GeoGebra to design sliders allows students to intuitively see the corresponding changes in the parabola's opening size, direction, vertex position, and axis of symmetry as parameters change. This experience is far superior to static analysis.

#### 4.4.2 Transition from "Demonstration" to "Exploration"

The higher value of information technology lies not in "teacher demonstration" but in "student exploration." Inquiry-based tasks can be designed: given a function form, have students adjust parameters themselves, observe the resulting graph changes, and generalize the role of parameters; given a graph feature, have students deduce the conditions the parameters must satisfy. This type of exploration activity transforms students from passive viewers into active discoverers, and the number-shape combination thought is naturally internalized during exploration.

It should be noted that technology is a tool, not the goal. Students should be consciously guided to express and verify their intuitive discoveries on the screen using mathematical language, avoiding the substitution of "intuition for proof."

#### 4.4.3 Integration of Information Technology and Traditional Blackboard Writing

Information technology and traditional blackboard writing each have their strengths and should complement each other. Software is suitable for displaying dynamic processes and complex graphics; blackboard writing is suitable for presenting reasoning steps and key conclusions. A reasonable teaching process could be: use software demonstration to elicit conjectures, then use blackboard writing for logical proof; or complete reasoning on the blackboard, then use software to verify the conclusions. This integration leverages the intuitive advantages of technology without losing the rigorous essence of mathematics.

### 4.5 Systematic Training and Evaluation of the Number-Shape Combination Thought

#### 4.5.1 Systematic Planning by Grade and Module

The penetration of the number-shape combination thought should have an overall plan, clarifying goals and key points for each grade level. The junior high school stage mainly focuses on "from shape to number," using intuitive graphics to establish algebraic concepts and initially cultivating drawing habits and graphic interpretation abilities; the senior high school stage strengthens "from number to shape" and "two-way conversion," achieving deep number-shape integration in modules such as functions, analytic geometry, and inequalities. For each specific unit, the penetration points, methods, and expected levels for number-shape combination in that unit should be clarified.

Teaching research groups can jointly develop a "Teaching Sequence for the Number-Shape Combination Thought," forming teaching guides for each grade level and module, avoiding isolated and arbitrary penetration.

#### 4.5.2 Classification of Levels for Number-Shape Combination Ability

The evaluation of number-shape combination ability should have hierarchical criteria. Referring to the SOLO taxonomy, the following levels can be distinguished:

**Level 1 (Prestructural):** Cannot establish a connection between numbers and shapes, solving problems relying entirely on a single representation method.

**Level 2 (Unistructural):** Can apply number-shape combination when prompted, but the application is mechanical and imitative.

**Level 3 (Multistructural):** Can actively apply number-shape combination to solve routine problems, but lacks flexibility and adaptability.

**Level 4 (Relational):** Can flexibly choose between numerical or graphical representation methods according to the characteristics of the problem and convert freely between them.

**Level 5 (Extended Abstract):** Can creatively apply number-shape combination to solve complex or non-routine problems and reflect on and abstract general methods.

#### 4.5.3 Combining Formative and Summative Evaluation

In terms of formative evaluation, methods such as classroom observation, learning logs, and analysis of incorrect answers can be used to understand the development status of students' number-shape combination ability and adjust teaching in a timely manner. Design specialized "Number-Shape Combination Practice" to evaluate the quality of drawing, accuracy of graphic interpretation, and flexibility of conversion.

In terms of summative evaluation, test papers should deliberately include questions that assess number-shape combination ability. These can be explicit assessments (explicitly requiring "draw the figure") or implicit assessments (difficult to solve without drawing a figure). The guiding role of evaluation can direct the attention of both teachers and students towards the teaching and training of the number-shape combination thought.

## 5. Conclusions and Recommendations

### 5.1 Main Conclusions

This study systematically explored the penetration strategies of the number-shape combination thought in middle school mathematics teaching, drawing the following main conclusions:

First, the number-shape combination is one of the most fundamental and important methods of thought in middle school mathematics, possessing multiple educational values such as reducing cognitive difficulty, cultivating thinking ability, and embodying the essence of mathematics.

Second, there are three major problems in current number-shape combination teaching: at the teacher level, cognitive deviations and single strategies; at the student level, weak graphic representation ability and lack of conversion awareness; at the teaching level, content segmentation and lack of systematic design.

Third, effective penetration of the number-shape combination thought requires a systematic strategy system, including: two-way construction in concept teaching, flexible application in problem-solving teaching, characteristic application in different modules, empowerment through information technology, and systematic training and evaluation support.

Fourth, the penetration of the number-shape combination thought is a long-term, systematic process, requiring teachers to have conscious awareness, systematic design, and sustained action.

### 5.2 Teaching Recommendations

Based on the conclusions of this study, the following recommendations are made for frontline teachers and teaching research work:

First, enhance one's own understanding of the number-shape combination thought. Teachers should deeply understand the essential connotation and educational value of number-shape combination, treating it as a fundamental thought running through teaching, not as scattered techniques.

Second, strengthen systematic instructional design. Teaching research groups should jointly develop a teaching sequence for the number-shape combination thought, clarifying the goals, penetration points, and evaluation criteria for each grade level and module.

Third, provide students with ample opportunities to "do mathematics." Let students draw figures themselves, discover quantitative relationships themselves, and attempt conversion of representation methods themselves, experiencing number-shape combination through "doing."

Fourth, value the appropriate use of information technology. Use dynamic geometry software as an exploration tool to guide students to actively explore the connection between numbers and shapes, while maintaining mathematical rigor.

Fifth, establish a diversified evaluation system. Incorporate number-shape combination ability into the scope of teaching evaluation, guiding the improvement of teaching and learning through the combination of formative and summative assessment.

### 5.3 Research Limitations and Future Directions

This study is primarily based on literature analysis and case summarization. The effectiveness of the strategies needs to be further tested in authentic teaching contexts. Subsequent research could employ action research or quasi-experimental research methods, implementing these strategies in classroom teaching and collecting empirical data to verify their effects. At the same time, the relationship between number-shape combination ability and other dimensions of mathematical core competencies (such as mathematical abstraction, logical reasoning, mathematical modeling) could be explored as a research direction, investigating the unique function of number-shape combination in fostering core competencies. Furthermore, developing scientific evaluation tools for number-shape combination ability is also a topic worthy of in-depth investigation.

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