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Original Paper

Liberal Paradox in Extended Framework

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Abstract

Since Sen (1970) made contribution on the liberal paradox, his impossibility theorem has been debated in the literature for a long time. Some economists criticized the Sen's formulation of rights and some economists had tried to find some resolution schemes to solve this impossibility theorem. Although the literature of this reseach lasts long time, we also think that there also exists research value in this field. In this paper, we will explore the rationality of Sen's impossibility theorem again. But we will give up the original framework used by Sen and use the extended framework which refers to individuals' preferences for social decision-making system. Then we will give an analysis of a resolution scheme proposed by Suzumura (1983) to testify whether this resolution scheme is still feasible in extended framework. Totally speaking, in this paper, we will concentrate on the extended framework and futher analyse the Sen's liberal paradox. So we will discuss more details in the later sections.

Keywords

social choice, impossibility results, liberal

1. Introduction

Since Sen put forward the *liberal paradox*, a lot of attentions have been paid for the paradox problem by economists. Because this impossibility theorem means that the Pareto optimal social system is not consistent with social system which should respect human rights. This outcome will lead people to doubt whether the current social system is correct. Therefore, many economists, sociologists and politicians criticized Sen' paradox in various ways. For example, some economists criticized the Sen's social choice theoretic formulation of rights (see for example Gaertner, W., Pattanaik, P. K. and Suzumura, K. (1992)). It was Nozick (1974) that does not agree with the idea that individual right should be visualized as the social choice theoretic formulation based on the individual's preference over social alternatives. Instead, he think that right should be the freedom of individual to select one option from some available options and individual's choice will fix some features of social states. Some economists such as Gibbard (1974), Suzumura (1983), Harel and Nitzan (1987) tried to find some resolution scheme to solve the paradox. But whatever they do, all of these economists had tried to overturn Sen's paradox.

In this paper we do not talk about the debate around the Sen's impossiblity theorem. Our objective of this paper is to use new extended framework to analyse the Sen's impossiblity theorem. About the extended framework, we can refer to some words from Pattanaik and Suzumura (1996):

Instead of working with the traditional informational basis which focuses on individual preference orderings over the set **X** of conventionally defined social alternatives, we propose to expand the conceptual structure so that the informational basis will consist of individual preference orderings over the Cartesian product of **X** and **M**, where **M** stands for the set of social decision-making mechanisms through which the actual soial choices are made.

Why do we need the extended framework? Because we think that people will have a different preference ordering when they face different social decision-making mechanism, even they will have the same social alternative from different social decision-making mechanism. This means that some people will be concequentialist and some people will be non-consequentialist. But in this paper, we only use the extended framework to analyse Sen's liberal paradox and could not refer more about consequentialism and non-consequentialism. Then we also must know that there exist many formulations of social decision-making mechanism, but we will use one of them in this paper and we will give the interpretation in the subsequent section.

The structure of this paper is as follows. In Section 2, we will give the analysis of Sen's Impossibility Theorem in Extended Framework, which consists of a little reivew of Sen' theorem in original framework and the checking of Sen' theorem in extended framework. Section 3 will refer to resolution schemes in extended framework. In this section, we also give some review of resolution schemes in original framework and then we will give a test of a original resolution scheme in extended framework. In Section 4, we will make a conclusion about this paper and give some questions of this paper and some possible research direction in the future.

2. Sen's Impossibility Theorem in Extended Framework

2.1 Review of Sen's Impossibility Theorem

As we know, Sen's concept of individual liberty is described in the social choice framework. So we must give some fundamental definition about framework. Let $N = \{1,2,\dots,n\}$ be set of all individuals in the society. Let X be the set of all social alternatives. For each individual $i \in N$, let us define a preference ordering $R_i \subseteq X \times X$ and together with each person's preference ordering we can form a Profile $\mathrm{R} = \{R_1, R_2, \dots R_n\} \in \mathcal{R}^n(X)$, where $\mathcal{R}^n(X)$ is the set of all logically possible profiles. Then we define a social choice rule which can be a function f that maps each profile $\mathrm{R} = \{R_1, R_2, \dots R_n\} \in \mathcal{R}^n(X)$ into a choice function $C^{\mathrm{R}} = f(\mathrm{R})$. Then we define the $C^{\mathrm{R}}(S)$, where S is a feasible set of social alternatives and $S \subseteq X$, that stands for the set of alternatives that are chosen from S under the profile $\mathrm{R} = \{R_1, R_2, \dots R_n\} \in \mathcal{R}^n(X)$. Let ψ stand for the family of all S and consists of all nonempty finite subsets of X. Because we can not predict which specific profile $\mathrm{R} = \{R_1, R_2, \dots R_n\} \in \mathcal{R}^n(X)$ will appear, and which specific $S \in \psi$ we will get, the first condition(Suzumura(2011), Chapter 23) must be proposed as follows.

Condition U: Unrestricted domain The social choice rule f must be able to determine a social choice function $C^{\mathrm{R}} = f(\mathrm{R})$ with the full domain ψ for each and every profile $\mathrm{R} = \{R_1, R_2, \dots R_n\} \in \mathcal{R}^n(X)$.

Then we will review the most important condition called **Sen's Minimal Libertarianism** which be described as follows (Suzumura (1996)).

Sen's Minimal Liberty: There are at least two persons such that for each of them there is at least one pair of social states over which he/she is decisive.

The intended meaning of this condition is illustrated by Sen as follows: "Given other things in the society, if you prefer to have pink walls rather than white, then the society should permit you to have this, even if a majority of the community would like to see your walls white. Similarly, whether you should sleep on your back or on your belly is a matter in which the society should permit you absolute freedom, even if a majority of the community is nosey enough to feel that you must sleep on your back" (1970).

Sen defined that, for each indiviudal $i \in N$ in the society, a respected private sphere $D_i \subseteq X \times X$, which means that for any pair $(x, y) \in D_i$, the only difference between x and y is i's private sphere. Then Sen defines the second condition below (Suzumura 2011) **Condition L: Libertarianism** For $i \in N$, $(x, y) \in D_i$ and given R_i, the social choice rule can be said to respect i's private sphere D_i if and only if

$(x, y) \in D_i \cap P(R_i) \Rightarrow [x \in S \Rightarrow y \notin C^{\mathrm{R}}(S)]$

for all $S \in \psi$, holds for all $\mathrm{R} \in \mathcal{R}^n(X)$, where $C^{\mathrm{R}} = f(\mathrm{R})$

Finally, Sen give the last condition below(Suzumura 2011)

Condition P: Pareto Principle For any $(x, y) \in X$, and any $\mathrm{R} \in \mathcal{R}^n(X)$

$$(x, y) \in \bigcap_{i \in N} P(R_i) \Rightarrow [x \in S \Rightarrow y \notin C^{\mathrm{R}}(S)]$$

for all $S \in \psi$, holds for all $\mathrm{R} \in \mathcal{R}^n(X)$, where $C^{\mathrm{R}} = f(\mathrm{R})$

Then together with three conditions above, Sen gives his famous theorem as follows.

Theorem 1. ***(Impossibility of a Paretian Liberal)*** *There exists no social choice rule f that satisfies the **U**, **L** and **P**.*

The Sen's impossibility theorem is too strong in social choice framework which be proposed by himself. In the next subsection, we will discuss it in extended framework.

2.2 Checking in Extended Framework

Then let us start the checking of Sen's theorem in Extended Framework. Firstly, we also let $N = \{1,2,\ldots,n\}$ be set of all individuals in the society and $X = \{x, y, z, \ldots\}$ be the set of all social alternatives.

Secondly, we will give the extended structure which consists of Catesian product of X and S, where $S = \{S_1,\ldots,S_m\}$ ($S_j \subseteq X$) stands for the family of the oppotunity set. For example, the pair (x, S_1) means that we can choose a social alternative x from opportunity S_1. Then we must define that $|X| > |S|$ to confirm that Sen's framework is not a special case of the extended framework. If we do not define this, we will have not any sense to check the Sen's impossibility theorem in extended framework. Then we let $Y = X \times S = \{(x, S_1),\ldots,(x,S_m),(y,S_1),\ldots,(y,S_m),\ldots\}$ be the set of all extended social alternatives. In Pattanaik & Suzumura(1996), the definition of extended framework is a little different. S is placed by G which stands for a decision procedure defined by a game form. This could refer to different fomulation of right, but in this papaer we still use Sen' fomulation of right. So we use the opportunity set to replace the game form. Then let $\psi = \{T_1,\ldots,T_p\}(T_l \subseteq Y)$ be the family of all feasible sets of extended social alternatives.

Thirdly, let us define a preference ordering $R_i \subseteq Y \times Y$ and together with each person's preference ordering we can form a Profile $\mathrm{R} = \{R_1,R_2,\ldots R_n\} \in \mathcal{R}^n(Y)$,where $\mathcal{R}^n(Y)$ is the set of all logically possible profiles. Then we define a social choice rule which can be a function f that maps each profile $\mathrm{R} = \{R_1,R_2,\ldots R_n\} \in \mathcal{R}^n(Y)$ into a choice function $C^{\mathrm{R}} = f(\mathrm{R})$. Then we also define the $C^{\mathrm{R}}(T)$ that stand for set of extended social alternatives that be chosen from T under the profile $\mathrm{R} = \{R_1,R_2,\ldots R_n\} \in \mathcal{R}^n(T)$. In the next step, we also need the **Condition U** as follows(Suzumura(2011), Chapter 23).

Condition U: Unrestricted domain The social chocie rule f must be able to determine a social choice function $C^{\mathrm{R}} = f(\mathrm{R})$ with the full domain ψ for each and every profile $\mathrm{R} = \{R_1,R_2,\ldots R_n\} \in \mathcal{R}^n(Y)$.

The contents of **Condition** L^* is a little different from Sen's. Let us define, for each individual $i \in N$, a respected private sphere in extended framework $D_i \subseteq Y \times Y$ and together each individual's respected private sphere in extended framework we can get a righs system $\mathrm{D} = \{D_1.D_2,\ldots,D_n\}$. Then we give the second condition below.

Condition L^*: Libertarianism in extended framework For $i \in N$, $((x,S_j),(y,S_j)) \in D_i$ and given R_i, the social choice rule can be said to respect i's private sphere D_i if and only if

$((x,S_j),(y,S_j)) \in D_i \cap P(R_i) \Rightarrow [(x,S_j) \in T \Rightarrow (y,S_j) \notin C^{\mathrm{R}}(T)]$

for all $T \in \psi$, holds for all $\mathrm{R} \in \mathcal{R}^n(Y)$, where $C^{\mathrm{R}} = f(\mathrm{R})$

Some people may ask why do we define the Libertarianism in extended framework in this way. We suppose that opportunity set is designed by some institutions or government and individuals in the society can not control opportunity set. So individuals in the society could only have decisiveness over the social alternatives which had better be in the same opportunity set if this pair of extended social alternative is in his/her respected private sphere.

Finally, let us give the third condition.

Condition P^*: Pareto Priciple in extended framework For any $(x,S_j),(y,S_k) \in Y$, and any

$\mathrm{R} \in \mathcal{R}^n(Y)$

$x = y$ or $x \neq y$, $S_j = S_k$ or $S_j \neq S_k$

$((x,S_j),(y,S_k)) \in \bigcap_{i \in N} P(R_i) \Rightarrow [(x,S_j) \in T \Rightarrow (y,S_k) \notin C^{\mathrm{R}}(T)]$

for all $T \in \psi$, holds for all $\mathrm{R} \in \mathcal{R}^n(Y)$, where $C^{\mathrm{R}} = f(\mathrm{R})$.

Then let we talk about whether Sen's impossibility theorem is also correct in the extended framework. Unfortunately, Sen's impossibility theorem is too strong to be overturn. So we also get a impossiblity theorem and the proof will be given after theorem.

Theorem 2. ***(Impossibility Theorem in extended framework)****There exists no social choice rule f that satisfies* **U**, L^* *and* P^*.

Before starting the proof of this theorem, we should give some important definition which will be often used in the subsequent proof.

Let us introduce the concept of Coherent rights system in extended framework which is similar to Farrell(1976), Suzumura(1978) and Suzumura(2011). **Definition. Coherent rights system in extended framework**: Let $\mathrm{D} = \{D_1 . D_2, \ldots, D_n\}$ be an n-tuple of subsets of $Y \times Y$. A critical loop in **D** is a sequence of ordered pairs $\{((x^\mu,S_j),(y^\mu,S_j))\}_{u=1}^{t} (t \geq 2)$ such that

$$((x^\mu,S_j),(y^\mu,S_j)) \in \bigcup_{i=1}^{n} D_i \quad \text{for all} \quad \mu \in \{1,2,\ldots,t\} \tag{1}$$

$$\text{there exists no } i^* \in \{1,2,\ldots,n\} \text{such that } ((x^\mu,S_j),(y^\mu,S_j)) \in D_{i^*} \text{ for all } \mu \in \{1,2,\ldots,t\}, and \tag{2}$$

$$x^1 = y^t \text{ and } (x^\mu,S_j) = (y^{\mu-1},S_j) \text{ hold true for all } \mu \in \{2,\ldots,t\}. \tag{3}$$

We say that $\mathrm{D} = \{D_1.D_2, \ldots, D_n\}$ is a coherent rights system if and only if there exists no critical loop in **D**.

Why do we propose the definition of **Coherent rights system in extended framework**? Because we can say that if **D** is not coherent, it is impossible to construct a collective choice rule that satisfies the **D** without using the **Pareto principle**. Suzumura has given the example to prove in Suzumura (1983) and here we will give the similar edition of proof in extended framework.

Proof. Let's suppose that the rights system $\mathrm{D} = \{D_1.D_2, \ldots, D_n\}$ contains a critical loop $\{((x^\mu, S_j), (y^\mu, S_j))\}_{u=1}^t (t \geq 2)$ that satisfies (1), (2), and (3). Then

$\exists$ a mapping $\kappa : \{1,2, \ldots, t\} \to \mathrm{N}$

$$\forall \mu \in \{1,2, \ldots, t\}: \left((x^\mu, S_j), (y^\mu, S_j)\right) \in D_{\kappa(\mu)} \tag{4}$$

$$\exists \mu^1, \mu^2 \in \{1,2, \ldots, t\}: \kappa(\mu^1) \neq \kappa(\mu^2) \tag{5}$$

Consider a profile $a = \{R_1^a, R_2^a, \ldots, R_n^a\} \in \mathcal{R}^n(Y)$ such that

$$\begin{aligned} &\left((x^1, S_j), (x^2, S_j)\right) \in P(R^a_{\kappa(1)}), \left((x^2, S_j), (x^3, S_j)\right) \in \\ &P(R^a_{\kappa(2)}), \ldots, \left((x^t, S_j), (x^1, S_j)\right) \in P(R^a_{\kappa(t)}) \end{aligned} \tag{6}$$

By (5) we can know that there is no contradiction in (6) with transitivity of $R_i^a (i \in N)$. Because we must require the collective choice rule to satisfy the righs system **D**, then from (4) and (6) we can have that

$$\begin{aligned} &\forall \mu \in \{1,2, \ldots, t-1\}: \left((x^\mu, S_j), (x^{\mu+1}, S_j)\right) \\ &\in P(R^a_{\kappa(\mu)}) \cap D_{\kappa(\mu)} \to (x^{\mu+1}, S_j) \notin C^a(T) \end{aligned} \tag{7}$$

and

$\left((x^t, S_j), (x^1, S_j)\right) \in P(R^a_{\kappa(t)}) \cap D_{\kappa(t)} \to (x^1, S_j) \notin C^a(T)$

where $C^a = F(a)$ and $T = \{(x^1, S_j), (x^2, S_j), \ldots, (x^t, S_j)\}$.

So we have that $C^a(T) = \emptyset$. Proof is over.

Then let us give the proof **Theorem 2**.

Proof. In this proof, we will follow the Suzumura's method (Suzumura (2011), Chapter 23) and according to definition we have proposed before. We will define that $|X| > 5$ and $S_j = \{x, y, z, w\}$ firstly to make sure that our proof make sense.

1 : D is coherent.

Suppose that $\left((x, S_j), (y, S_j)\right) \in D_l$ and $\left((z, S_j), (w, S_j)\right) \in D_k$, where $(x, S_j) \neq (y, S_j)$ and $(z, S_j) \neq (w, S_j)$. Because D_l and D_k is symmetric, we also have $\left((y, S_j), (x, S_j)\right) \in D_l$ and $\left((w, S_j), (z, S_j)\right) \in D_k$.

1.1:

$\{(x,S_j),(y,S_j)\} = \{(z,S_j),(w,S_j)\}$

If $(x,S_j) = (w,S_j)$ and $(y,S_j) = (z,S_j)$, there exists a critical loop in **D** of order 2. Because **D** is assumed to be coherent, this is an obvious contradiction. The situation that $(x,S_j) = (z,S_j)$ and $(y,S_j) = (w,S_j)$ is impossible.

1.2:

$\{(x,S_j),(y,S_j)\} \cap \{(z,S_j),(w,S_j)\} = \{(z,S_j)\}$ or $\{(w,S_j)\}$

Suppose that $(y,S_j) = (z,S_j)$ and consider a profile $\mathrm{R}^1 = (R_1^1, R_2^1, \ldots, R_n^1) \in \mathcal{R}^n(Y)$ and $T = \{(x,S_j),(y,S_j),(w,S_j)\}$. Then suppose that $((x,S_j),(y,S_j)) \in D_l \cap P(R_l^1)$, $((y,S_j),(w,S_j)) \in D_k \cap P(R_k^1)$ and $((w,S_j),(x,S_j)) \in P(R_i^1)$ for all $i \in N$.

So this profile is reasonable by **Condition U**. We now define that

$C^{\mathrm{R}^1}((x,S_j),(y,S_j),(w,S_j))$, where $C^{\mathrm{R}^1} = f(\mathrm{R}^1)$. **Condition** L^* syas that

$(y,S_j),(w,S_j) \notin C^{\mathrm{R}^1}((x,S_j),(y,S_j),(w,S_j))$, whereas **Condition** P^* says that

$(x,S_j) \notin C^{\mathrm{R}^1}((x,S_j),(y,S_j),(w,S_j))$. Thus, $C^{\mathrm{R}^1}((x,S_j),(y,S_j),(w,S_j)) = \emptyset$, which is a contradiction.

1.3:

$\{(x,S_j),(y,S_j)\} \cap \{(z,S_j),(w,S_j)\} = \emptyset$

Consider a profile $\mathrm{R}^2 = (R_1^2, R_2^2, \ldots, R_n^2) \in \mathcal{R}^n(Y)$ and $T = \{(x,S_j),(y,S_j),(w,S_j),(z,S_j)\}$. Then suppose that $((x,S_j),(y,S_j)) \in D_l \cap P(R_l^2)$, $((z,S_j),(w,S_j)) \in D_k \cap P(R_k^2)$ and $((w,S_j),(x,S_j)), ((y,S_j),(z,S_j)) \in P(R_i^2)$ for all $i \in N$.

So this profile is reasonable by **Condition U**. We now define that

$C^{\mathrm{R}^2}((x,S_j),(y,S_j),(z,S_j),(w,S_j))$, where $C^{\mathrm{R}^2} = f(\mathrm{R}^2)$. **Condition** L^* says that

$(y,S_j),(w,S_j) \notin C^{\mathrm{R}^2}((x,S_j),(y,S_j),(z,S_j),(w,S_j))$, whereas **Condition** P^* entails that

$(x,S_j),(y,S_j) \notin C^{\mathrm{R}^2}((x,S_j),(y,S_j),(z,S_j),(w,S_j))$. Thus,

$C^{\mathrm{R}^2}((x,S_j),(y,S_j),(z,S_j),(w,S_j)) = \emptyset$, which is a contradiction.

2: D is incoherent

If **D** is not coherent, there must exist a critical loop $\{((x^{\mu},S_j),(y^{\mu},S_j))\}_{\mu=1}^{t}(t \geq 2)$ and also there exists a mapping $\kappa: \{1,2,\ldots,t\} \rightarrow \mathrm{N}$. Then they will satisfies

$$((x^{\mu},S_j),(y^{\mu},S_j)) \in D_{\kappa(\mu)} \quad \text{for all } \mu \in \{1,2,\ldots,t\} \tag{8}$$

and

$$\kappa(\mu^1) \neq \kappa(\mu^2) \quad \text{for some } \mu^1,\mu^2 \in \{1,2,\ldots,t\} \tag{9}$$

and let's consider a new profile $\mathrm{R}^3 = (R_1^3,R_2^3,\ldots,R_n^3) \in \mathcal{R}^n(Y)$ such that

$$((x^1,S_j),(x^2,S_j)) \in P(R^3_{\kappa(1)}),((x^2,S_j),(x^3,S_j)) \in P(R^3_{\kappa(2)}),\ldots,$$
$$((x^t,S_j),(x^1,S_j)) \in P(R^3_{\kappa(t)})$$

By (2), we can conclude that R^3 satisfies the transitivity of R_i^3 for all $i \in N$. So it must satisfies **Condition U**. Suppose that $T = \{(x^1,S_j),(x^2,S_j),\ldots,(x^t,S_j)\}$. From $((x^{\mu},S_j),(x^{\mu+1},S_j)) \in P(R^3_{\kappa(\mu)}) \cap D_{\kappa(\mu)}$, we have that $(x^{\mu+1},S_j) \notin C^{\mathrm{R}^3}(T)$ for all $\mu \in \{1,2,\ldots,t-1\}$, whereas $((x^t,S_j),(x^1,S_j)) \in P(R^3_{\kappa(t)}) \cap D_{\kappa(t)}$ implies that $(x^1,S_j) \notin C^{\mathrm{R}^3}(T)$. Thus, $C^{\mathrm{R}^3}(T) = \emptyset$, which is a condiction.

From the proof above, we can conclude that Sen's impossibility theorem still supported even in the extended framework. But we cannot say that the extended framework make no sense. Because Sen's impossibility theorem is too strong. We can also can check some resolution schemes in this extended framework and this will be done in the next section.

3. Resolution Schemes in Extended Framework

3.1 Review of Resolution Schemes of Sen's Impossibility Theorem

Since Sen proposed his liberal paradox, many economists tried to give some resolution schemes. Because there exists too many resolution schemes in the literature, in this paper we will review two famous schemes of them briefly.

The first resolution scheme is called Voluntary Alienation of Conferred Libertarian Rights which was proposed by Gibbard (1974). Gibbard proposed that if the individual in the society exercise his right and unfortunately get a worse outcome than what he get under the situation in which he does not exercise his right, this individual will waive his right. So Gibbard give a definition of **Waiver set** (Suzumura (2011)) as follows

Definition. Waiver set: Suppose that the profile $\mathrm{R} = \{R_1,R_2,\ldots R_n\} \in \mathcal{R}^n(X)$ and $S \in \psi$ are such that $x \in S$ and for $i \in N$ $(x,y) \in D_i \cap P(R_i)$. Then i has the right to execise his right to eliminate y from $C^{\mathrm{R}}(S)$, but he will think that he should waive his right if exists a sequence $\{y_1, y_2, \dots, y_\lambda\}$ in S such that

$$y_\lambda = x, (y, y_1) \in R_i \& y \neq y_1, \tag{10}$$

and

$$(\forall t \in \{1,2,\dots,\lambda-1\}) : (y_t, y_{t+1}) \in (\bigcap_{j \in N} P(R_j)) \cup (\bigcup_{j \in N - \{i\}} [D_j \cap P(R_j)]). \tag{11}$$

Then $W_i(\mathrm{R}|S)$ which is subset of D_i will be called the **Waiver set** of $i \in N$ at $(\mathrm{R}|S)$ when $(x, y) \in W_i(\mathrm{R}|S)$ if and only if (10) and (11) hold for some sequence $\{y_1, y_2, \dots, y_\lambda\}$ in S.

Then Gibbard proposed his new **Condition GAL** as follows

Condition GAL: Gibbard's Alienable Libertarianism For any profile $\mathrm{R} = \{R_1, R_2, \dots R_n\} \in \mathcal{R}^n(X)$, any opportunity set $S \in \psi$, any $i \in N$, and any $x, y \in X$, if $(x, y) \in D_i \cap P(R_i)$ and $(x, y) \notin W_i(\mathrm{R}|S)$, then $[x \in S \Rightarrow y \notin C^{\mathrm{R}}(S)]$, where $C^{\mathrm{R}} = f(\mathrm{R})$.(Suzumura(2011), Chapter 23)

Then Gibbard gave his possibility theorem as follows

Theorem 3. ***(Pareto-Consistency of Gibbard's Alienable Libertarianism)*** *There exists a social choice rule f that satisfies **U**, **P** and **GAL**. (For the proof see Gibbard (1974, pp.401-402)*

Gibbard's resolution scheme is not perfect as his proof described in his paper. Some economists such as Kelly (1976) criticized his definition of **Waiver set** (For details see Suzumura (2011), 622-623), so we will not choose Gibbard's resolution scheme to check in extended framework.

Among the resolution schemes which have been proposed, the most parts of them criticized the original definition of **Libertarianism** and kept **Pareto principle** intact. But the resolution scheme which was proposed by Sen (1976, 1982, Chapter 14) and Suzumura (1978, 1983, Chapter 7) made accusing axe on the **Pareto Pinciple**. Because they think that some situation we faced behind **Pareto Pinciple** sounds like a meddlesome action.

To illustrate the resolution scheme, let us quote one monologue wrote by Suzumura ((2011), Chapter 23) as follows

"What a terrible necktie he is wearing! He would look much nicer if he were to wear something more decent. However, I am keenly aware that to choose a nectie of his own is his personal matter and none of my business. Thus, my meddlesome concern—no matter how strong it may be—should not count in socially deciding what necktie he should wear".

Sen-Suzumura called such individual who has this monologue *a liberal individual*. The main purpose of this resolution scheme is that if there exists at least one liberal individual in the society, even though in the society there exists many meddlesome individuals, we also can protect the liberty.

Referring to (Suzumura (2011), Chapter 23), for fomulizing the liberal individual, Sen-Suzumura consider a profile $\mathrm{R} = \{R_1, R_2, \dots R_n\} \in \mathcal{R}^n(X)$ and let R_i^0 which is a transitive subrelation of

R_i for $i \in N$ be the preference ordering which i wants to be counted in social choice. For example, if $(x, y) \in P(R_i)$, this means that i prefers x to y, but i need not this preference to be counted in the society and if $(x, y) \in P(R_i^0)$, this means that i prefers x to y and wants this preference to be counted in the social choice.

Then Sen-Suzumura gave the new version of the **Pareto Principle**.

Condition CP: Conditional Pareto Principle For all $x, y \in Y$ and $\mathrm{R} \in \mathcal{R}^n(X)$

$(a)(x,y) \in R_N^0 \Rightarrow [x \in S/C^{\mathrm{R}}(S) \Rightarrow y \notin C^{\mathrm{R}}(S)] and$

$(b)(x,y) \in P(R_N^0) \Rightarrow [x \in S \Rightarrow y \notin C^{\mathrm{R}}(S)]$

Where $R_N^0 : \bigcap_{i \in N} R_i^0$ and $C^{\mathrm{R}} = f(\mathrm{R})$

Then Suzumura gives the definition of R_i^0. But before giving it, Suzumura gives a **Coherence Lemma**(Suzumura 1983) as follows.

Lemma 1. *There exists an ordering extension R of Q: =* $\bigcup_{i \in N}(D_i \cap R_i)$ *if and only if the rights-system **D** is coherent. (For the proof see Suzumura (1983))*

Then given any profile $\mathrm{R} \in \mathcal{R}^n(X)$ and $E(\mathrm{D}{:}\mathrm{R})$ which is the set of all ordering extension of Q, Suzumura defines the R_i^0 of liberal individual in the society by $R_i^0 = R \cap R_i$ for some $R \in E(\mathrm{D}{:}\mathrm{R})$ and the R_i^0 of illiberal individual in the society by $R_i^0 = R_i$, which means that the liberal individual in the society only insist on his parts of preferences which is consistent with other individual's preferences and the illiberal individual in the society insist on all of his preferences regardless of other's preferences.

Then Suzumura-Sen give a possibility theorem as follows (Sen (1976), Suzumura (1983))

Theorem 4. ***(****Possibility Theorem in the Presence of a Liberal Individual) There exist a rational social choice rule with **U**, which satisfies **CP** and realizes the initially conferred coherent rights-system **D** with respect to the profile* R^0*, if there exists at least one liberal individual in the society.(For the proof refer to Suzumura(1983).)*

3.2 Liberal Individual in Extended Framework

Because we are interested in what effect the liberal individual will bring in extended framework, in this subsection, we will try to check that whether the resolution scheme proposed by Sen (1976) and Suzumura (1983) is also feasible in extended framework we used in this paper.

Firstly, we must give the new **Conditional Pareto Principle** in extended framework as follows

Condition CP^*: For all $(x, S_j), (y, S_j) \in Y$ and $\mathrm{R} \in \mathcal{R}^n(Y)$

$(a)((x,S_j),(y,S_j)) \in R_N^0 \Rightarrow [(x,S_j) \in T/C^{\mathrm{R}}(T) \Rightarrow (y,S_j) \notin C^{\mathrm{R}}(T)] and$

$(b)((x,S_j),(y,S_j)) \in P(R_N^0) \Rightarrow [(x,S_j) \in T \Rightarrow (y,S_j) \notin C^{\mathrm{R}}(T)]$

Where $R_N^0 : \bigcap_{i \in N} R_i^0$ and $C^{\mathrm{R}} = f(\mathrm{R})$

For this new version of **Conditional Pareto Principle**, some people will doubt on why the opportunity set in this condition is the same. From the definition of $R_N^0 := \bigcap_{i\in N} R_i^0$, we know that $R_N^0 \subset R_i^0$ where i is liberal. From definition $R_i^0 = R \cap R_i$ for some $R \in E(\mathrm{D}|\mathrm{R})$ we have introduced in Section 2.2, we know that $R_N^0 \subset R$ which R is ordering extension of $Q := \bigcup_{i\in N}(D_i \cap R_i)$. Obviously we can conclude that the opportunity set in this new condion must be same. But before we get this conclusion, we had better prove that Q has an ordering extension R, that is, we must prove that the **Coherence Lemma in extended framework** is correct.

Then let us give the **Coherence Lemma in extended framework** which is almost the same as original Lemma proposed by Sen-Suzumura.

Lemma 2. ***(in extended framework)*** *There exists an ordering extension R of each and every $D_i \cap R_i (i \in N)$ for every profile $a = \{R_1, R_2, ... R_n\}$ $a \in \mathcal{R}^n(Y)$ if and only if the rights system $\mathrm{D} = \{D_1 . D_2, ..., D_n\}$ in extended framework is coherent.*

Proof. We will follow the method used by Suzumura (1983)

At first, let us prove the 'if' part. Let **D** be coherent, and take any profile $a = (R_1^a, R_2^a, ..., R_n^a) \in \mathcal{R}^n(Y)$.

Let Q_i^a be defined by

$Q_i^a = D_i \cap R_i^a (i \in N)$

Let Q^a be defined by

$$\mathrm{Q}^a = \bigcup_{i\in N} Q_i^a$$

Firstly let us prove that Q^a is consistent. Why must we do this step? Because if Q^a is consistent, there exists an ordering extension R^a of Q^a by Theorem (Suzumura (1983), Chapter 1, 16-17) (Note 1)

Let us suppose, to the contrary, that there exists a set

$\{(x^1, S_j), (x^2, S_j), ..., (x^t, S_j)\} \in \psi$

that satisfies $\big((x^1, S_j), (x^2, S_j)\big) \in P(\mathrm{Q}^a)$, $\big((x^\mu, S_j), (x^{\mu+1}, S_j)\big) \in \mathrm{Q}^a$ for $\mu \in \{2,3, ..., t-1\}$ and $\big((x^t, S_j), (x^1, S_j)\big) \in \mathrm{Q}^a$. From $\big((x^1, S_j), (x^2, S_j)\big) \in P(\mathrm{Q}^a)$, by the definition of asymmetric part we have that $\big((x^1, S_j), (x^2, S_j)\big) \in \mathrm{Q}^a$ and $\big((x^2, S_j), (x^1, S_j)\big) \notin \mathrm{Q}^a$. Then we can have that $\big((x^1, S_j), (x^2, S_j)\big) \in \mathrm{Q}_i^a$ for some $i \in N$ and $\big((x^2, S_j), (x^1, S_j)\big) \notin \mathrm{Q}_i^a$ for all $i \in N$ by the definition of Q^a. So we will have that $\big((x^1, S_j), (x^2, S_j)\big) \in P(\mathrm{Q}_i^a)$ for some $i \in N$. There exists a mapping $\kappa: \{1,2, ..., t\} \to N$

such that $\big((x^1,S_j),(x^2,S_j)\big) \in P(\mathrm{Q}^a_{\kappa(1)})$, $\big((x^\mu,S_j),(x^{\mu+1},S_j)\big) \in P(\mathrm{Q}^a_{\kappa(\mu)})$ for all $\mu \in \{2,3,\ldots,t-1\}$, and $\big((x^t,S_j),(x^1,S_j)\big) \in P(\mathrm{Q}^a_{\kappa(t)})$.

So we have that

$(a)\{\kappa(1),\kappa(2),\ldots,\kappa(t)\}$ is not a singleton set

Why do we have this conclusion? Suppose, to the contrary, it is a singleton set. Then we have that $\big((x^1,S_j),(x^2,S_j)\big) \in P(\mathrm{Q}^a_i)$, $\big((x^\mu,S_j),(x^{\mu+1},S_j)\big) \in P(\mathrm{Q}^a_i)$ for all $\mu \in \{2,3,\ldots,t-1\}$, and $\big((x^t,S_j),(x^1,S_j)\big) \in P(\mathrm{Q}^a_i)$ for one person $i \in N$ so that by the definition of Q^a_i, we have that $\big((x^1,S_j),(x^2,S_j)\big) \in R^a_i$, $\big((x^\mu,S_j),(x^{\mu+1},S_j)\big) \in R^a_i$ for all $\mu \in \{2,3,\ldots,t-1\}$, and $\big((x^t,S_j),(x^1,S_j)\big) \in R^a_i$ for one person $i \in N$, which is in contradiction with transitivity of R^a_i.

$(b)\big((x^\mu,S_j),(x^{\mu+1},S_j)\big) \in D_{\kappa(\mu)}$ $\mu \in \{1,2,\ldots,t-1\}$ and $\big((x^t,S_j),(x^1,S_j)\big) \in D_{\kappa(t)}$

Therefore, by (a) and (b), we can say that D contains a critical loop, which is a contradiction with which we have defined before. So Q^a is consistent and there must exists an ordering extension R^a of Q^a.

Then we have $Q^a_i \subset \mathrm{Q}^a \subset \mathrm{R}^a (i \in N)$ and $P(\mathrm{Q}^a) \subset P(\mathrm{R}^a)$. So if we can prove that $\forall i \in N$: $P(\mathrm{Q}^a_i) \subset P(\mathrm{Q}^a)$ is correct, then we succeed. So suppose that $\exists i \in N$, any $(x,S_j),(y,S_j) \in Y$ such that $\big((x,S_j),(y,S_j)\big) \in P(Q^a_i)$, and $\big((y,S_j),(x,S_j)\big) \in Q^a_{i'}$ for some $i' \in N$. Then by the definition of Q^a_i, we can say that **D** contains a critical loop which is a contradiction.

Secondly, let us prove the 'only if' part. Let us suppose, to the contrary, that **D** is not coherent. Then this will mean that we have a critical loop in **D**. So, there exists a sequence $\{(x^\mu,S_j)\}^t_{\mu=1}$ such that $\big((x^1,S_j),(x^2,S_j)\big) \in D_{\kappa(1)},\ldots,\big((x^t,S_j),(x^1,S_j)\big) \in D_{\kappa(t)}$ for some mapping $\kappa\colon\{1,2,\ldots,t\} \to N$. Then let a profile $a = (R^a_1,R^a_2,\ldots,R^a_n) \in \mathcal{R}^n(Y)$ which satisfies that

$$\begin{aligned}&\big((x^1,S_j),(x^2,S_j)\big) \in P(Q^a_{\kappa(1)}),\big((x^2,S_j),(x^3,S_j)\big) \in Q^a_{\kappa(2)},\ldots,\\ &\qquad\big((x^t,S_j),(x^1,S_j)\big) \in Q^a_{\kappa(t)}\end{aligned} \tag{12}$$

where $Q_i^a = D_i \cap R_i^a (i \in N)$. Because $\{\kappa(1), \kappa(2), \ldots, \kappa(t)\}$ is not a singleton set, we will satisfy the transitivity of $R_i^a (i \in N)$ in (12). Then let R^a be an ordering extension of all $Q_i^a (i \in N)$ in common. Then (12) will lead to a outcome that

$$\left((x^1, S_j), (x^2, S_j)\right) \in \mathrm{R}^a, \left((x^2, S_j), (x^3, S_j)\right) \in \mathrm{R}^a, \ldots,$$
$$\left((x^t, S_j), (x^1, S_j)\right) \in \mathrm{R}^a$$

which is in contradiction with transitivity of R^a. Proof is over.

Then the next step is to test that whether in extended framework, original possibility theorem will change into the impossibility theorem. Fortunately, this original possibility theorem is still correct by our proof subsequently. Then let us give the new version of **Theorem 4**.

Theorem 5. ***(Possibility Theorem in the Presence of Liberal Individual in extended framework)***: *There exists a rational social choice rule with* ***U****, which satisfies CP^* and realizes the initially conferred coherent rights-system* ***D*** *in extended framework with respect to the restricted preference profile R^0, if there exists at leaset one liberal individual in the society.*

Proof. In this proof, we will follow the method used by Suzumura (1983)

Let N_1 be the set of all liberal individuals and $\mathrm{N}_1 \subset \mathrm{N}$. Then take any profile $a = (R_1^a, R_2^a, \ldots, R_n^a) \in \mathcal{R}^n(Y)$ and let $\mathcal{R}^a$ be the set of all ordering extensions of

$$\mathrm{Q}^a = \bigcup_{i \in N} D_i \cap R_i^a$$

Let R_i^{0a} be

$$R_i^{0a} = \begin{cases} \exists R^i \in \mathcal{R}^a : R_i^a \cap R^i & \text{if } i \in \mathrm{N}_1 \\ R_i^a & \text{otherwise} \end{cases}$$

and let R_0^a be defined by

$$\mathrm{R}_0^a = \{\left((x, S_j), (y, S_k)\right) \in Y \times Y \mid \text{ when } S_k = S_j, \left((y, S_k), (x, S_j)\right) \notin \mathrm{P} \cup \mathrm{P}(\mathrm{R}_N^{0a}) \quad \text{when } S_k \neq S_j, \left((y, S_k), (x, S_j)\right) \notin \mathrm{P}_1\} \tag{13}$$

where $\mathrm{P} = \bigcap_{i \in N_1} P(R^i), \mathrm{R}_N^{0a} = \bigcap_{i \in N} R_i^{0a}$ and $\mathrm{P}_1 = \bigcap_{i \in N} P(R_i^a)$

Then let us prove that R_0^a is complete, reflexive and acyclic.

(1) R_0^a is complete as well as reflexive.

Assume, to the contrary, that there exist $(x, S_j), (y, S_k) \in Y$ such that

$\left((x, S_j), (y, S_k)\right) \notin \mathrm{R}_0^a$ and $\left((y, S_k), (x, S_j)\right) \notin \mathrm{R}_0^a$

When $S_k = S_j$, then we obtain

$\left((x, S_j), (y, S_k)\right) \in \mathrm{P} \cup \mathrm{P}(\mathrm{R}_N^{0a})$ and $\left((y, S_k), (x, S_j)\right) \in \mathrm{P} \cup \mathrm{P}(\mathrm{R}_N^{0a})$

Then we have four cases below.

(a) $\left((x, S_j), (y, S_k)\right) \in \mathrm{P}, \left((y, S_k), (x, S_j)\right) \in \mathrm{P}$

(b) $\big((x,S_j),(y,S_k)\big) \in \mathrm{P}(\mathrm{R}_N^{0\alpha}), \big((y,S_k),(x,S_j)\big) \in \mathrm{P}(\mathrm{R}_N^{0\alpha})$

(c) $\big((x,S_j),(y,S_k)\big) \in \mathrm{P}, \big((y,S_k),(x,S_j)\big) \in \mathrm{P}(\mathrm{R}_N^{0\alpha})$

(d) $\big((x,S_j),(y,S_k)\big) \in \mathrm{P}(\mathrm{R}_N^{0\alpha}), \big((y,S_k),(x,S_j)\big) \in \mathrm{P}$

(a) and (b) contradict the asymmetry of **P** and the asymmetry of $\mathrm{P}(\mathrm{R}_N^{0\alpha})$. Then consider (c) and (d). Let take any $i_0 \in N_1$. We can have that

$\mathrm{P} \subset \mathrm{P}(R^{i_0})$ and $\mathrm{P}(\mathrm{R}_N^{0\alpha}) \subset R_{i_0}^{0\alpha} \subset R^{i_0}$

So we can say that (c) and (d) contradict the ordering of R^{i_0}.

When $S_k \neq S_j$, then we obtain

$\big((x,S_j),(y,S_k)\big) \in \mathrm{P}_1$ and $\big((y,S_k),(x,S_j)\big) \in \mathrm{P}_1$

This contradict the asymmetry of P_1.

So we have proved that R_0^{α} is complete as well as reflexive.

(2) The next step is to establish

$$\text{For } S_j = S_k,\ \mathrm{P}(\mathrm{R}_0^{\alpha}) = \mathrm{P} \cup \mathrm{P}(\mathrm{R}_N^{0\alpha}) \tag{14}$$

$$\text{For } S_j \neq S_k,\ \mathrm{P}(\mathrm{R}_0^{\alpha}) = \mathrm{P}_1 \tag{15}$$

For $S_j = S_k$, if $\big((x,S_j),(y,S_k)\big) \in \mathrm{P}(\mathrm{R}_0^{\alpha})$, then $\big((y,S_k),(x,S_j)\big) \notin \mathrm{R}_0^{\alpha}$ which implies $\big((x,S_j),(y,S_k)\big) \in \mathrm{P} \cup \mathrm{P}(\mathrm{R}_N^{0\alpha})$ by definition. Therefore, $\mathrm{P}(\mathrm{R}_0^{\alpha}) \subset \mathrm{P} \cup \mathrm{P}(\mathrm{R}_N^{0\alpha})$. To show the converse, suppose that there exists an ordered pair $\big((x,S_j),(y,S_k)\big) \in Y \times Y$ such that $\big((x,S_j),(y,S_k)\big) \in \mathrm{P} \cup \mathrm{P}(\mathrm{R}_N^{0\alpha})$, but $\big((x,S_j),(y,S_k)\big) \notin \mathrm{P}(\mathrm{R}_0^{\alpha})$, so that $\big((y,S_k),(x,S_j)\big) \notin \mathrm{R}_0^{\alpha}$ and $\big((x,S_j),(y,S_k)\big) \notin \mathrm{P}(\mathrm{R}_0^{\alpha})$. But this contradicts the completeness or reflexivity of R_0^{α}. Therefore (14) is valid.

For $S_j \neq S_k$, if $\big((x,S_j),(y,S_k)\big) \in \mathrm{P}(\mathrm{R}_0^{\alpha})$, then $\big((y,S_k),(x,S_j)\big) \notin \mathrm{R}_0^{\alpha}$ which implies $\big((x,S_j),(y,S_k)\big) \in \mathrm{P}_1$ by definition. Therefore, $\mathrm{P}(\mathrm{R}_0^{\alpha}) \subset \mathrm{P}_1$. To show the converse, suppose that there exists an ordered pair $\big((x,S_j),(y,S_k)\big) \in Y \times Y$ such that $\big((x,S_j),(y,S_k)\big) \in \mathrm{P}_1$, but $\big((x,S_j),(y,S_k)\big) \notin \mathrm{P}(\mathrm{R}_0^{\alpha})$, so that $\big((y,S_k),(x,S_j)\big) \notin \mathrm{R}_0^{\alpha}$ and $\big((x,S_j),(y,S_k)\big) \notin \mathrm{P}(\mathrm{R}_0^{\alpha})$. But this contradicts the completeness and reflexivity of R_0^{α}. Therefore (15) is valid.

We now prove that R_0^{α} is acyclic. If there exists a set $\{(x^1,S_j^1),(x^2,S_j^2),\ldots,(x^t,S_j^t)\} \in \psi$, such that

$\big((x^{\mu},S_j^{\mu}),(x^{\mu+1},S_j^{\mu+1})\big) \in \mathrm{P}(\mathrm{R}_0^{\alpha})$ for all $\mu \in \{1,2,\ldots,t-1\}$

and

$\left((x^t,S_j^t),(x^1,S_j^1)\right) \in \mathrm{P}(\mathrm{R}_0^a)$

If $S_j^t \neq S_j^1$, we can easily at leaset derive that

$\left((x^1,S_j^1),(x^t,S_j^t)\right) \in \mathrm{P}_1 \subset P(R_i^a)(i \in N),$

$\left((x^t,S_j^t),(x^1,S_j^1)\right) \in \mathrm{P}_1 \subset P(R_i^a)(i \in N).$

Then we arrive at a contradiction with the transitivity of R_i^a. If $S_j^t = S_j^1$, we can also easily at leaset derive that

$\left((x^1,S_j^1),(x^t,S_j^t)\right) \in R^i(i \in N_1),$

$\left((x^t,S_j^t),(x^1,S_j^1)\right) \in R^i(i \in N_1).$

Then we arrive at a contradiction with the transitivity of R^i.

So R_0^a is acyclic.

Now R_0^a is complete, reflexive and acyclic.

$$\forall T \in \psi\colon C^a(T) = G(T,\mathrm{R}_0^a) \tag{16}$$

defines a rational choice function C^a on a choice space (Y,ψ). By this C^a with the profile $a \in \mathcal{R}^n(Y)$, we have a rational collective choice rule.

Then we must prove that this rule satisfies **Condition CP(a)**, suppose that there exist $(x,S_j),(y,S_j) \in Y$ such that

$\left((x,S_j),(y,S_j)\right) \in \mathrm{R}_N^{0a}$, $(x,S_j) \in T\backslash C^a(T)$,

and $(y,S_j) \in C^a(T)$ for some $T \in \psi$.

We can give a contradiction from this supposition by proving that

$\left((x,S_j),(z,S_k)\right) \in \mathrm{R}_0^a$ is true for all $(z,S_k) \in T$

so that $(x,S_j) \in C^a(T)$

should be true in view of (16). Therefore take any $(z,S_k) \in T$,

(1) For $S_k \neq S_j$.

Because $(y,S_j) \in C^a(T)$ is assumed before, we have $\left((z,S_k),(y,S_j)\right) \notin \mathrm{P}_1$. By the definition of P_1, we have $\left((z,S_k),(y,S_j)\right) \notin \mathrm{P}_1$ if and only if $\left((y,S_j),(z,S_k)\right) \in R_i^a$ for some $i \in N$. By assumption, $\left((x,S_j),(y,S_j)\right) \in \mathrm{R}_N^{0a}$, so that we have $\left((x,S_j),(y,S_j)\right) \in R_i^{0a} \subset R_i^a$ for this $i \in N$. Because R_i^a is transitive, $\left((x,S_j),(y,S_j)\right) \in R_i^a$ and $\left((y,S_j),(z,S_k)\right) \in R_i^a$ imply that

$\left((z,S_k),(x,S_j)\right) \notin P(R_i^a)$

Then

$$\left((z,S_k),(x,S_j)\right) \notin \mathrm{P}_1 \tag{17}$$

(2) For $S_k = S_j$.

Because $(y, S_j) \in C^a(T)$ is assumed before, we have $((z, S_k), (y, S_j)) \notin \mathrm{P} \cup \mathrm{P}(\mathrm{R}_N^{0a})$, so that
$((z, S_k), (y, S_j)) \notin \mathrm{P}$ and $[((z, S_k), (y, S_j)) \in \mathrm{R}_N^{0a} \vee ((y, S_j), (z, S_k)) \in \mathrm{R}_N^{0a}]$
By definition of **P**, we have $((z, S_k), (y, S_j)) \notin \mathrm{P}$ if and only if $((y, S_j), (z, S_k)) \in R^i$ for some $i \in N$. By assumption, $((x, S_j), (y, S_j)) \in \mathrm{R}_N^{0a}$, so that we have $((x, S_j), (y, S_j)) \in R_i^{0a} \subset R^i$ for this $i \in N$. Because R^i is transitive, $((x, S_j), (y, S_j)) \in R^i$ and $((y, S_j), (z, S_k)) \in R^i$ imply that
$((z, S_k), (x, S_j)) \notin P(R^i)$
Then

$$((z, S_k), (x, S_j)) \notin \mathrm{P} \tag{18}$$

Suppose that
$((z, S_k), (y, S_j)) \notin \mathrm{R}_N^{0a}$
which implies
$((z, S_k), (y, S_j)) \in R_i^{0a}$ for some $i \in N$
If $i \in N \backslash N_1$, we have that
$((y, S_j), (z, S_k)) \in P(R_i^a)$,
Follows from
$((x, S_j), (y, S_j)) \in \mathrm{R}_N^{0a}$,
we have that
$((x, S_j), (z, S_k)) \in P(R_i^a)$;
that is,
$((z, S_k), (x, S_j)) \notin R_i^a$
Therefore,
$((z, S_k), (x, S_j)) \notin \mathrm{R}_N^{0a}$
If $i \in N_1$,
$[((z, S_k), (y, S_j)) \notin R_i^a \vee ((z, S_k), (y, S_j)) \notin R^i]$,
that is,
$[((y, S_j), (z, S_k)) \in P(R_i^a) \vee ((y, S_j), (z, S_k)) \in P(R^i)]$,
In view of $((x, S_j), (y, S_j)) \in \mathrm{R}_N^{0a} \subset (R^i \cap R_i^a)$, we have that
$[((x, S_j), (z, S_k)) \in P(R_i^a) \vee ((x, S_j), (z, S_k)) \in P(R^i)]$,
Then we have
$[((z, S_k), (x, S_j)) \notin R_i^a \vee ((z, S_k), (x, S_j)) \notin R^i]$.
Hence
$((z, S_k), (x, S_j)) \notin R_i^{0a} \Rightarrow ((z, S_k), (x, S_j)) \notin \mathrm{R}_N^{0a}$

Therefore, in every case, we have that $((z,S_k),(x,S_j)) \notin \mathrm{R}_N^{0a}$. Because $\mathrm{P}(\mathrm{R}_N^{0a}) \subset \mathrm{R}_N^{0a}$, we can say that

$$((z,S_k),(x,S_j)) \notin \mathrm{P}(\mathrm{R}_N^{0a}) \tag{19}$$

By (17), (18), (19), we have that

For $S_j \neq S_k$, $((z,S_k),(x,S_j)) \notin \mathrm{P}_1$

For $S_j = S_k$, $((z,S_k),(x,S_j)) \notin \mathrm{P} \cup \mathrm{P}(\mathrm{R}_N^{0a})$

So

$((x,S_j),(z,S_k)) \in \mathrm{R}_0^a$.

Then **CP(a)** is satisfied.

Let us start the proof of **CP(b)**. Suppose that there exist $(x,S_j),(y,S_j) \in Y$ and $T \in \psi$, such that

$((x,S_j),(y,S_j)) \in \mathrm{P}(\mathrm{R}_N^{0a})$, $(x,S_j) \in T$, and $(y,S_j) \in C^a(T)$

Then we have

$((y,S_j),(x,S_j)) \in \mathrm{R}_0^a$,

so that

$((x,S_j),(y,S_j)) \notin \mathrm{P} \cup \mathrm{P}(\mathrm{R}_N^{0a})$

But this contradicts with $((x,S_j),(y,S_j)) \in \mathrm{P}(\mathrm{R}_N^{0a})$.

Finally, we will prove that the collective choice rule satisfies the rights system **D**. Consider a contrary example that there exist $i \in N$ and $T \in \psi$ that satisfies

$((x,S_j),(y,S_j)) \in D_i \cap P(R_i^a)$, $(x,S_j) \in T$, and $(y,S_j) \in C^a(T)$.

Then we have that

$((y,S_j),(x,S_j)) \in \mathrm{R}_0^a \Rightarrow ((x,S_j),(y,S_j)) \notin \mathrm{P} \cup \mathrm{P}(\mathrm{R}_N^{0a})$

By the definition of $\mathcal{R}^a$, we have that

$P(\mathrm{Q}^a) \subset \bigcap_{i \in N_1} P(R^i) = \mathrm{P}$,

so that if we can prove that

$$D_i \cap P(R_i^a) \subset P(\mathrm{Q}^a) \tag{20}$$

we success, because then we have that

$((x,S_j),(y,S_j)) \in D_i \cap P(R_i^a) \subset \mathrm{P}$

which contradicts

$((x,S_j),(y,S_j)) \notin \mathrm{P} \cup \mathrm{P}(\mathrm{R}_N^{0a})$

Then we start to prove (20). Let

$((w,S_j),(z,S_j)) \in D_i \cap P(R_i^a)$.

Then

$$\big((w,S_j),(z,S_j)\big) \in \mathrm{Q}^a,$$

so that if

$$\big((w,S_j),(z,S_j)\big) \notin P(\mathrm{Q}^a)$$

we have that

$$\big((z,S_j),(w,S_j)\big) \in \mathrm{Q}^a = \bigcup_{i \in N} Q_i^a$$

Then there exists $l \in N\,(l \neq i)$ such that

$$\big((z,S_j),(w,S_j)\big) \in D_l \cap R_l^a$$

So we can say that **D** has a critical loop, which is a contradiction of the assumed coherence of **D**.

After our proof of this new theorem, we can say that the resolution scheme proposed by Sen-Suzumura is an extremely strong scheme which can be used in the original framework and also can be applied in the extended framework in this paper.

4. Concluding Remarks

In this paper we proposed a new extended framework to analyse the Sen's liberal paradox again. We emphasize that people make a preference ordering not only by the outcomes they achieve but also by some procedures or some opportunity sets from which they get this outcome, so we make use of an information basis which consists of social alternatives and social decision-making system to define this new framework. Unfortunately, we cannot but say that Sen's impossibility theorem is an extremely theorem and we can not overturn it even though we use the new extended framework. Then we want to try to analyse that the originial resolution scheme proposed by Sen (1976) and Suzumura (1983) in the extended framework. Fortunately, we succeed. This resolution scheme still withstands the test of extended framework proposed in this paper.

But we also can find some problems we are facing. The first one is that in the Libertarianism^* we define that only when the opportunity set must be same and contains x and y simultaneously, the individual $i \in N$ has decisiveness over the pair of $\big((x,S_j),(y,S_k)\big)$. Why do we define that the opportunity set must be same? I will give explaination about it. Because I supposed that opportunity set is designed by some institutions or government and individuals in the society only can exercise their rights when they face the same social policy (opportunity set) which be set by government. In other words, the procedures of exercising a right is that firstly, some institutions or government set a policy (opportunity set), then facing with the policy, individuals in the society will choose to exercise or not to exercise his right. But there must exist some people who do not agree with my idea. They think that this condition is too rigorous and they think that if $S_j \subset S_k$ or $S_k \subset S_j$ and S_j, S_k all contains x and y, or the case in which $x, y \in S_j \cap S_k$, the individual also have the decisiveness over the pair of $((x, S_j), (y, S_k))$ even the opportunity set is designed by some institutions or government. This problem actually deserves us to research in the future, because if we do this we must test Sen's paradox problem again and probably get a different outcome.

The second one is that we also can replace the opportunity set with some other social decision-making system. For example, we can use some procedures by which we can get the social alternative. This also probably can get a different outcome against in this paper.

The last one is that we can propose that the individual in the society which consists of the individual who is consequentialism and the individual who is non-consequentialism which we have referred in the **Introduction**. So we can consider a more complicated situations. This is research agenda in the future we will face.

In the future, I prepare to pay my attention to the definition of social decision-making mechanism. Some people have doubt whether the opportunity set is suitable to be a social decision-making mechanism and I also have given the reason why they doubt. Now I have two directions. One is that I can change the definition of $Liberalism^*$ to find a more perfect explanation of definition and still use the opportunity set as social decision-making mechanism. The other one is that I will find one more suitable definition of social decision-making mechanism and give a reasonable explanation. I think that this is hard and now I have no inspiration. But I believe that in the near future I will give a surprise.

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Notes

Note 1. A binary relation R on X has an ordering extension if and only if it is consistent (For proof see Suzumura (1983), Chapter 1, 16-17).