Journal of Research in Philosophy and History
Vol. 3, No. 2, 2020
www.scholink.org/ojs/index.php/jrph
ISSN 2576-2451 (Print) ISSN 2576-2435 (Online)

Original Paper

Dialectical Infinity and the Third Mathematical Crisis

—On the Fundamental Error of Actual Infinity

Zhang Hong[1]

[1] Bank of China, China

Received: August 31, 2020 Accepted: September 10, 2020 Online Published: September 17, 2020

doi:10.22158/jrph.v3n2p73 URL: http://dx.doi.org/10.22158/jrph.v3n2p73

Abstract

This paper discusses the problem of finity and infinity based on the philosophical perspectives of opposing idealism and receiving dialectical materialism. Based on Hegel's dialectical infinity view, this paper makes a comprehensive criticism of the thought of actual infinity. After Hegel's dialectical infinite thought scientifically explained the limit concept in calculus, the Second Mathematical Crisis caused by the contradiction of infinitesimal quantity was solved thoroughly. However, the mathematics world has not learned the experience and lessons in history, has always adhered to the idealist thought and methodology of actual infinity, this thought finally brought the third crisis to mathematics. At the end of this paper, based on the infinite view of dialectical materialism, the author analyzes the Principle of Comprehension and the Maximum Ordinal Paradox, and points out that the essence of the Principle of Comprehension is a kind of actual infinity thought. Only by limiting the Principle of Comprehension to a potential infinity can we solve the Third Mathematical Crisis completely.

Keywords

actual infinity, real infinity (genuine infinity,true infinity), bad infinity (wrong infinity), the infinite view of dialectical materialism, Infinite Exchange Paradox, the Principle of Comprehension

1. The Infinite Exchange Paradox Reveals an Emperor's New Clothes: Infinity Has Never Been Completed

The so-called *Infinite Exchange Paradox* refers to the idea that we use the thought of *actual infinity*—that is, the idea that the infinite process can be accomplished, and we can transform the two equivalent infinities (with *one-to-one correspondence*) into mutually nonequivalent infinities. This profoundly exposes the inherent defects of *the actual infinite thought*, it moves the contradiction to infinity, but the contradiction never disappears. In other words, the infinite process is impossible to complete, thus further supporting *the infinite view of dialectical materialism*.

For a detailed description of the *Infinite Exchange Paradox*, please read the second section of *PHILOSOPHICAL INFINITY AND MATHEMATICAL INFINITY* (see Zhang Hong, Zhuang Yan, 2019).

2. Hegel's Dialectical View of Infinity. *Being* is not Equivalent to Completion

Hegel, the great philosopher, made a deep and detailed study of the infinite and infinite things from philosophy and Mathematics. Its core idea is to admit the existence of *bad infinity*, and think that *the bad infinity* is not the true infinity; that we should give up and go beyond the *infinite progress*, and then realize the essence of infinity, by sublating *bad infinite* to understand *the real infinity*. At the same time, he also believes that these two kinds of infinite are mutually transformed, with unity, that infinite has two kinds of infinite attributes. However, there is an essential difference between Hegel's *real infinity* and *actual infinity* in Mathematics. Therefore, Hegel's infinite view is essentially a kind of dialectical infinite view.

2.1 The Essence of Hegel's Thought of Bad Infinity and Real Infinity

A detailed exposition of Hegel's infinite thought has been published in the third section of my article *PHILOSOPHICAL INFINITY AND MATHEMATICAL INFINITY* (see Zhang H. & Zhuang Y., 2019).

As for Hegel's dialectical view of infinity, we conclude as follows: The mutual stipulation of finite things makes up infinity (that is, the *real infinity*), infinity is the unity of *bad infinity* and *real infinity*, and it is a *free self Being*; *Real infinity* cannot be separated from *bad infinity*, *Being-for-self* is inseparable from *Being-in-itself*. Human understanding of the infinite, from possible to reality, from the abstract to the concrete, has completed the transformation from *bad infinity* to *real infinity*. The *real infinity* is present, concrete, completed infinite, is *Being-for-self* and rational *Being*, is the completed quality; and the *bad infinity* is possible, abstract, uncompleted infinite, is *Being-in-itself* and *intellectual Being*.

The difference between *real infinity* and *bad infinity* reflects the opposition between dialectical understanding and metaphysical understanding. *Real infinity* and *bad infinity* are the basic forms of infinity. Hegel put forward the profound dialectical conclusion that *real infinity* contains and subtracts *bad infinity*, and tries to grasp infinity concretely and realistically, and opposes abstract inferences about it. The transformation from *bad infinity* to *real infinity* is the transformation from understanding to reason, is a great leap in human understanding of infinity, and is the highest task of Hegel's hilosophy

2.2 Engels Highly Affirmed Hegel's Dialectical Infinite Thought

Engels thought that infinity was a kind of objective *being*, affirmed Hegel's dialectical infinite view. Engels thinks that infinity is purely composed of finity, and that this contradiction will not be destroyed, that is, the infinite process can not be completed. This is a direct negation of the *actual infinity*, but also a direct affirmation of Hegel's view of *bad infinity*.

Engels thought that infinite things can be known, but also can not be known, which is consistent with the essence of thought of Hegel's *bad infinity* and *real infinity*. That is, the *real infinity* (as the universal connection between infinite things, the inner connection) can be known and completed, and the *bad infinity* can not be known, can not be completed.

For more details, see Section 4 of my article, *Philosophical Infinity and Mathematical Infinity* (see Zhang H. & Zhuang Y., 2019).

2.3 The Difference between Actual Infinity and Hegel's Dialectical Infinity

(1) The main features and relations of Hegel's *bad infinity* and *real infinity*:

- The *bad infinity* and the *real infinity* are two aspects of the same infinity, two attributes, who also cannot do without whom. *Bad infinity* is the carrier of *real infinity*, *bad infinity* contains *real infinity*, *real infinity* is the intrinsic essence of infinity, and *bad infinity* is the external appearance of infinity.
- The infinite is the objective being, this is self-evident. The objective material world in infinite space, time is a typical infinite object.
- *Bad infinity:* admitting infinite objective existence, but that the *infinite process* is not to be completed, so it is a kind of *bad infinity*.
- *Real Infinity:* The view that infinity is *self-related in the other*, is the sublation of the *bad infinity*, is the common nature, the inner connection or universality of the infinite things. Therefore, it is considered that the infinity is knowable and can be accomplished. It is the result of a change in quality, is the sublation and negation of the *bad infinity*, is the other side of *bad infinity*, and thus is a must, is a more advanced finity.

(2) The main difference between *bad Infinity* and *potential infinity*:

Although people often regard the two as the same thing, but in philosophy, the two have a fundamental qualitative difference. The *potential infinity* view holds that infinity is not an objective being, but a potential, because it cannot be completed. The *bad infinity* view holds that infinity is an objective being, but at the same time, it can not be completed. So they have the same thing: infinite progress can't be done.Therefore, *potential infinity* can be regarded as a fragment of *bad infinity*, a segment of continuous development and change, a quantity of motion change, and a quantity that continually manifests bad infinity.

(3) The similarities and differences between *potential infinity* and *actual infinity*:

Combined with the mathematical and philosophical analysis of Zhu Wujia's views on these two kinds of infinity, the following points are summarized:

- On the objective existence of infinite: The *actual infinity* holds that the infinite objective exists, and the *potential infinity* thinks that infinity is not a kind of reality, and denies the existence of infinite objective;
- About the infinite whether can be accomplished: The *actual infinity* thinks that infinity can be done, and on the other hand, the *potential infinity* holds that infinite affirmatively cannot be completed.

In the viewer of *potential infinity*, infinity is a kind of potential, which is in constant construction, so it is not a kind of reality, this is obviously denying the objective existence of infinity with the impossibility of the subjective world's understanding of infinity. The infinite object as the objective thing is true, not with the will of human beings. Thus the concept of *potential infinity* has made a subjective mistake, so it is an infinite view of idealism.

(4) The main difference between *real infinity* and *actual infinity:*

After careful study of Hegel's infinite theory, it is obvious that the concept of Hegel's *real infinity* is completely different from that of *actual infinity* in Mathematics.

- *Actual infinity* view in mathematics: The infinite is an objective being, the *infinite progress* can be completed, and this is directly opposed to the *bad infinity* in philosophy, to admit *actual infinity* is a direct negation of *bad infinity*;
- Hegel's *real infinity*: refers to the internal relations, common essence and regularity of infinite things, and not whether an *infinite progress* can be completed. Since *real infinity* and *bad infinity* are dialectical and unified, it is necessary to admit that *real infinity* must admit the being of infinite objectivity, admit *bad infinity* and admit that it is impossible to complete the infinite.
- Common of the two: They all recognize the objective existence of infinite objects; they are a kind of finite (*the actual infinity* is a kind of finity, see behind the discussion).
- The difference between the two: *The actual infinity* thinks that *infinite progress* can be completed, and the *real infinity* thinks *infinite progress* impossible. The former is an idealist Epistemology, and the latter is a dialectical materialist Epistemology.

It can be seen that the concept of *actual infinity* runs counter to Hegel's thought of *bad infinity* and *real infinity.*

To sum up, *actual infinity* and *potential infinity* are all to make the finity and infinity opposition, but failed to unify them, in fact, this is the continuation of Kant's transcendental thought of antinomy. While Hegel's view of the infinity is the dialectical unity of the finity and infinity, combining quantity of the infinity and quality of the infinity to examine the infinity, thus forming a dialectical infinite view, which is *the infinite view of dialectical materialism.*

2.4 The Philosophical Essence of Actual Infinity Thought—A Comprehensive Critique of Actual Infinity Thought

The concept of *actual infinity* abandons *the contradiction between finity and infinity*, artificially assuming that it is purely on the other side of the infinite process, artificially severing the connection between finity and infinity, which is an idealism concept of infinity out and out, which is essentially a concept of finity. The existence of infinity is not equal to infinite completion, and incompleteness is the fundamental attribute of infinity.

(1) The *actual infinity* opposes finity and infinity, and makes itself unconnected with *infinite process*, so it is a false infinity.

The concept of *actual infinity* is a direct negation of *potential infinity*, and there is no certainty. In this way, *the contradiction between finity and infinity* (infinite process) disappears in it, and the contradiction is eliminated, and that is the end of infinity. Hegel has already given a clear explanation of *the contradiction between finity and infinity* from the dialectical point of view: "Now *the infinite progress* is only the *expression* of this contradiction, not its *resolution*" (Hegel, 1969, p. 227). Hegel also strongly refuted the dualism about the opposition between finity and infinity, and argues that the opposite infinity is not real infinity, but only finity, and the concept of *actual infinity* in mathematics is the contrary dualism. Therefore, *actual infinity* is not real infinity, but only a finite one.

We can't find any shadow of the *potential infinity* (*infinite progress* or *bad infinity*) in the *actual infinity*, but only the finite. This infinity, which has completely abandoned *the infinite process*, is not so much an infinite, as it is a real finite. Because it abandoned the *infinite process*, it also abandoned *the contradiction between finity and infinity*, so that they become a completely finity.

(2) Hegel emphasizes the unity of finity and infinity, and *the view of actual infinity* does not combine the infinite quantity with the infinite quality. It is in the negative of the quantity at the same time did not find the quality of things from the negative, but the quality of things with a negative, or even to the opposite of finite being, become an abstraction, artificially cut off contact between the infinite and finite, finally let oneself fall into nihilism, let oneself become an other that has no inner relation with the infinite process, that is to say, *actual infinity* and *infinite process* are entirely different things.

In Hegel's view, the *real infinity*, is *self-related in the other*, which is the common essence of the infinite things, the mutual stipulation, the internal connection or the universality, is negation of negation, but rather a kind of affirmation, is a kind of *Being-for-self*. At the same time, *finite being* is provided with *determinateness of quality*, is finite and variable, both positive and negative contains, and this kind of *determinateness of quality* (i.e., mutual regulation, common essence) is the link of finite and infinite, namely the link of *bad infinity* and *real infinity*. In the problem of Hilbert Hotel, *the view of actual infinity* in the negative of *infinite progress* at the same time denies the mutual stipulation (the inner relation) of the infinite things, the essence of *no spare room*, and concludes that *there is a spare room* in this completely opposite and contrary conclusion. This is completely contrary to the unity of finite and infinite, contrary to the unity of *bad infinity* and *real infinity*. The view of *actual infinity* believes such links do not exist, to deny its own regression (i.e., total negation rather than dialectical negation), see themselves as something that is purely on the other side of the world, no internal connection with the *infinite progress* (*bad infinity*), thus artificially cut off the relation between finite and infinite, eventually led to the no unified quality (there is no unity between the quality of the *actual infinity* and the quality of *infinite progress*), lead to the quantity of their own and their own quality is not uniform, and let the *actual infinity* become a thing that has nothing to do with the *infinite*

progress and the *potential infinity*, and becomes an abstract thing, an other. Hegel used *the line*, *the circle* to describe the *bad infinity*, *real infinity*, and *the actual infinity* is neither the *straight line*, not *the circle*, but it means that there is an *ultimate goal* (or *absolute truth*), and that the *ultimate goal* is a goal that can be reached after experiencing and completing an *infinite progress*. Therefore, it is not a dialectical sublation of *infinite progress* (i.e., *bad infinite*), but rather a mechanical, metaphysical and artificial abandonment, artificially cut off contact with *infinite progress*, so as to allow themselves to become something that is purely on the other side of the world. The qualitative change of things does not mean that things and the original things absolutely have no connection, completely cut off. On the contrary, the dialectical negation is the link of connection, the qualitative change with no connection does not exist. The reachability of finite space is often used to illustrate the completeness of the infinite process. For example, by using 1 of the coordinate axis X up to 0, people think that the infinite process of $\left\{\frac{1}{n}\right\}$ can be completed. The essence is to discard the infinite progress and replace the infinity with the finite, treat the infinity simply as a finite, impose the product of human thinking on the objective material, artificially create irreconcilable oppositions, while the infinite progress $\left\{\frac{1}{n}\right\}$ still exists independently. This kind of imaginary infinity (that is, *actual infinity*) is not infinite progress itself, but has no internal connection with infinite progress; it is just a spectator and other thing.

Hegel wants to combine the quantity of infinite with the quality of infinite, and holds that the quantitative infinity is not the infinite progress of simple quantity, but the stipulation of its own quality. (The so-called *quantum* is the quantity that has the stipulation, has the limit, this stipulation is the boundary.) But this kind of *actual infinite* in mathematics does not combine the quantity and the quality, therefore, the thought is followed by Kant's transcendental, subjective, metaphysical infinite thought, not Hegel's dialectical thought of infinite. Therefore, Hegel strongly criticized this kind of *actual infinity*: he considers that this kind of *actual infinity* is a nihilistic abstract, and it mistakenly regards the objective existence of infinity as the completed, already known infinity, treats the infinite simply as finite. In the analysis of Kant's thought of the antinomy he pointed out the essence of this kind of *actual infinity*, "Such intellect commits the error of holding such mental fictions, such abstractions, as an infinite number of parts, to be something true and actual; but this sensuous consciousness does not let itself be brought beyond the empirical element to thought. The Kantian solution of the antinomy likewise consists solely in the supposition that reason should not soar beyond sensuous perception and should take the world of appearance, the phenomental world, as it is. This solution leaves the content of the antinomy itself on one side; it does not attain to the nature of the *Notion* of its determinations, each of which, isolated on its own, is null and is in its own self only the transition into its other, the unity of both being *quantity* in which they have their truth." (Hegel, 1969, p. 199). As a result, the *actual infinity* becomes an abstraction, it is not only beyond *the infinite progress (bad infinity)*, also crossed

the real infinity, so as to let itself become a real *nothing*; because in Hegel's view, over the limit will be no.

(3) Infinity Exists and Cannot Be Completed

It completely replaces the objective material world itself with the subjective world product, or completely equates and imposes the subjective world product on the objective material world, which fundamentally negates the difference and contradiction between the subjective world and the objective world. The *actual infinity* abandoned *the contradiction between finity and infinity*, also abandoned *the real infinity*; it applies the product of human thought to the infinite world of objective material, and is opposed to the objective material world, thus the concept of *actual infinity* made the same mistake with Hegel, it is an out and out of idealism infinite view.

"Infinity can be done", this is a kind of understanding of the subjective world, but the view of *actual infinity* holds that it is the objective material world itself. Engels early has gaven a relentless criticism to this idealistic understanding. He said, "To attempt to prove the reality of any product of thought by the identity of thinking and being was indeed one of the most absurd delirious fantasies of—a Hegel" (Engels, 1947, p. 61). The above discussion shows that, we cannot prove the reality of our thinking products - *the infinite process can be completed*, because of the identity of the finite subjective world and the objective material world. This actually pointed out the fatal defect of the view of *actual infinity*. That is to say, *the infinite transcendence of thinking* does not mean *the completion of the objective infinite process*, and *the infinite transcendence of thinking* has never really been done. The *existent infinity* is not equal to the *known infinity*.

Infinity as infinite object itself, that is, as an *infinite process*, as a *bad infinity*, is impossible to complete, impossible to end. However, this does not mean that the infinite object (including both the completed finity and the unfinished infinity) cannot be regarded as a whole, a collection, and in fact, the human body always consciously, unconsciously put the infinite object as a whole, a collection. And this is the side of the transcendence of infinity, but also the natural, inevitable result and product of thought when the infinity is in the human brain, is the representation and reflection of the objective infinite world in the subjective finite world, and real infinity is the result of this transcendence. It is just that infinity cannot be fully expressed in the finite human consciousness, and it is because of the inability to fully express, this is the fundamental idea of *bad infinity* or *potential infinity*.

The concept of *actual infinity* always confuses the two concepts—infinite process (that is, *the bad infinity*) and global infinity (which is a kind of *real infinity*), confuses the finite and the infinite, confuses *bad infinity* and *real infinity*, and replaces the *infinite process* (which is the objective process of understanding the infinite material world) with the concept of global infinity (that is, the infinite transcendence of thinking, which is the concept of subjective understanding of the objective infinite world), that is, to replace *bad infinity* with *real infinity*, regards the transcendental cognition of the subjective finite world to the infinite object as the infinite process itself, regards the objective being of global infinity as the completion of the *infinite process*, regards the product of thinking as the process

of human beings understanding the objective infinite world itself, replaces objective things themselves with subjective cognition, so that *the contradiction between finity and infinity* is completely denied, completely abandoned, completely eliminated. In the view of *actual infinity*, *the contradiction between finity and infinity* is eliminated, so that it is possible to complete and end an *infinite process*; but *the contradiction between finity and infinity* has never been eliminated, and it exists forever. As Engels in *Anti-Dühring* sharply pointed out, "the removal of the contradiction would be the end of infinity. Hegel saw this quite correctly, and for that reason treated with well-merited contempt the gentlemen who subtilised over this contradiction." (Engels, 1947, p. 48) As a result, the concept of *actual infinity* is always confronted with a fatal blow, and constantly encounters new contradictions and new paradoxes, which is exactly the error of the infinite view of Cantor.

Just based on this wrong idealism recognition, in the view of *actual infinity*, when the infinite is a collection of objects, that is, as a whole, it can be done, because at this time the essence of *infinite* (that is, *the infinite process*, *the bad infinity*) has been completely abandoned, so it can be completed (that is, the infinite transcendence of thinking). The concept of *actual infinity* regards the completion as the *"infinite process can be completed"*, and the infinite objective being is equivalent to the *infinite process*, regards the completion of the *real infinity* as the completion of the *bad infinity*, with *real infinity* instead of *bad infinity*, thus completely abandon *the contradiction between finity and infinity*, this is the essence of the error of the *actual infinite* view. *Being* is a kind of objective, *completion* is a kind of subjective, and the *actual infinite* view confused the two, which is bound to be the opposite of materialism, will inevitably lead to insurmountable contradictions. Let us further illustrate: "a line" as a *whole infinite* is a kind of objective being, so it is a must, is *Being-for-self*, is the completed quality, is a finite (that is, a real infinity), but the infinite extension of the line is a *bad infinity*, which is abstract, cannot be completed, so is *Being-in-itself* and *intellectual being*; the *actual infinity* equated the former to the latter, and used the former to replace the latter, used the *Being* to replace the *completion*, thereby allowing itself to suffer a fatal blow.

This infinite concept, abandoned the *infinite process*, imposed the product of thinking on the objective material world, must be an infinite view anti dialectical materialism, is a real idealist view. However, Hegel's infinite view is not only to see *"the universal contact of the objective world (real infinity)"*, see the objective reality of *"the infinite process cannot be completed (bad infinity)"*, so it is a scientific concept of infinite, is a dialectical view of infinity, and therefore is an infinite view of dialectical materialism. Therefore, the infinite as an *infinite process* is eternal, is absolute, and cannot be completed; and when the infinite as a whole, as a *real infinite*, that is, to abandon the *infinite process*, this infinity can be completed, because at this time, this infinite is not so much an infinite, as it is a *real finity*. Such as the concept of natural numbers set N, is an infinite object, but it abandoned the *infinite process* between elements (*bad infinity*). The infinite object, which has left the *infinite process*, has become a mere concept or name, and has become a finite thing, just like a finite object in a meta-language. Thus, Russell is so sure to say, "We have to admit that there are real infinite sets," and

in fact, our mathematicians do, and do well. We consider the existence of the natural number set as the actual needs of people, which is the concrete embodiment of the dialectical infinite view.

In *Anti Dühring*, Engels fully criticized the absurd idea that *"the product of thought is imposed on the external objective world"* as Mr. Dühring, which effectively reveals the essence of mathematics, and holds that mathematics must reflect the objective material world correctly. He said, "But it is not at all true that in pure mathematics the mind deals only with its own creations and imaginations. The concepts of number and figure have not been derived from any source other than the world of reality. …… Like the idea of number, so the idea of figure is borrowed exclusively from the external world, and does not arise in the mind out of pure thought. ……Like all other sciences, mathematics arose out of the needs of men…… But, as in every department of thought, at a certain stage of development the laws, which were abstracted from the real world, become divorced from the real world, and are set up against it as something independent, as laws coming from outside, to which the world has to conform." (Engels, 1947, pp. 55-56).

Engels also gave a strong criticism on this idealism of Hegel's *absolute spirit*. He said, "The Hegelian system, in itself, was a colossal miscarriage—but it was also the last of its kind. It was suffering, in fact, from an internal and incurable contradiction. Upon the one hand, its essential proposition was the conception that human history is a process of evolution, which, by its very nature, cannot find its intellectual final term in the discovery of any so-called absolute truth. But, on the other hand, it laid claim to being the very essence of this absolute truth. A system of natural and historical knowledge, embracing everything, and final for all time, is a contradiction to the fundamental laws of dialectic reasoning. This law, indeed, by no means excludes, but, on the contrary, includes the idea that the systematic knowledge of the external universe can make giant strides from age to age." (Engels, 1947, p. 37). Here, Engels deeply criticized Hegel's idea of *absolute truth*, which is of *actual infinity*.

What we are here to oppose is the *final truth* of Mr. Dühring's Transcendentalism, which is criticized by Mr. Engels *in Anti-Dühring*; the concept of *actual infinity* is exactly the same thing as Transcendentalism, which in essence believes that the development of the world, the movement will have an end. While the essence of infinite essentially means the *inexhaustible attribute of infinity*, "inexhaustible" is "infinity", "infinity" is "inexhaustible". The *actual infinite* view holds that the infinity can be completed, which is directly opposed to the objective material world.

In *Anti Dühring*, Engels further criticized this idea, "If at any time in the development of mankind such a final, conclusive system of the interconnections within the world—physical as well as mental and historical—were brought about, this would mean that human knowledge had reached its limit, and, from the moment when society had been brought into accord with that system, further historical development would be cut short – which would be an absurd idea, sheer nonsense." (Engels, 1947, p. 53). "Mankind therefore finds itself faced with a contradiction: on the one hand, it has to gain an exhaustive knowledge of the world system in all its interrelations; and on the other hand, because of the nature both of men and of the world system, this task can never be completely fulfilled." (Engels, 1947,

p. 53). “Each mental image of the world system is and remains in actual fact limited, objectively by the historical conditions and subjectively by the physical and mental constitution of its originator. But Herr Dühring explains in advance that his mode of reasoning is such that it excludes any tendency to a subjectively limited conception of the world. We saw above that he was omnipresent—on all possible celestial bodies. We now see that he is also omniscient. He has solved the ultimate problems of science and thus nailed boards across the future of all science.” (Engels, 1947, p. 54)

The *existent* infinity is not equal to the *known infinity*, and the objective infinity is not equal to the subjective infinity, and the *actual infinity* is neither *bad infinity* nor *real infinity.* We can make a very vivid metaphor for the *real infinity* and the *bad infinity*: *the bad infinite* as a devil trapped in a closed box, if we don’t open this box, the devil will never come out (i.e., infinte progress can never be completed). In this analogy, *“the devil can never come out”* is a kind of *bad infinity*, and the *real infinity* refers to the intrinsic quality of the event—*“There’s a devil in the closed box”*. Combining the two, the statements of the whole event is that: *“there is a devil in the closed box, but the devil can never come out.”* The former indicates the nature of the event, the latter indicates that the event movement, the possibility of development, both interdependent but not the same. More vivid metaphor, we can take *the straight line* which Hegel used to describe the *bad infinity* to specify: *a point on a straight line moves freely in the straight line*. In this event, the *“point in a straight line”* represents the essence of the event, the inherent qualitative stipulation, reflecting the relationship between the point and the line, is a *real infinity*. And *the free movement of points in the straight line* represents the *bad infinity*, indicating that the *point* can reach any position on the straight line, but in any case the freedom movement cannot get rid of this line, are unlikely to change the intrinsic nature of the point in the line. If the expression of these events is still a bit fuzzy, then let us change the *“straight line”* into a closed *circle* to describe the event, the event will be transformed into such an event: A point on a closed circle moves freely on the circle. In the same way, *“the point in the closed circle”* represents the internal stipulation of the event, which reflects the relationship between the *point* and *circle*, represents the topological property of the event, and thus is a *real infinity*. And the *“free movement of a point on the circle”* represents the *bad infinity*, indicating that the *point* can move freely on the circle, but in any case it is impossible to get rid of the circle. That is to say, the result of *bad infinity* is *real infinity*, not *actual infinity*; in this case, we cannot find any shadow of *actual infinity*. *Actual infinity* thinks that it has already got rid of the *bad infinity*, but still in the *infinite progress*. It is not from the *quality* to consider the infinite, to sublate the *infinite progress*, but subjective, wishful thinking to interrupt the *infinite progress*, with subjective instead of objective, so it is a self deceiving, idealist epistemology, is a form of antinomy.

We can see everywhere Mr Dühring’s view of *“being is to be known”* which Engels criticized and opposed. If you say *existent infinity* once be thought, it is thought of as the *unified thing—accomplished infinity* (*actual infinity*), it is too funny. In *Anti Dühring*, Engels continues to discuss, “This is indeed system-creating! Within the space of the next six lines Herr Dühring has

transformed the *oneness* of being, by means of our unified thought, into its *unit.* As the essence of all thought consists in bringing things together into a unity, so being, as soon as it is conceived, is conceived as unified, and the idea of the world as indivisible; and because *conceived* being, the *idea of the world*, is unified, therefore real being, the real world, is also an indivisible unity" (Engels, 1947, p. 60). "To attempt to prove the reality of any product of thought by the identity of thinking and being was indeed one of the most absurd delirious fantasies of—a Hegel" (Engels, 1947, p. 61). This has been explained that: the *existent infinity* once we are thinking, we can't get the reality of this thinking product—*accomplished infinity* (*actual infinity*). Here, Engels ruthlessly refuted this idealism thought of *actual infinity* which escapes from *the contradiction between finity and infinity*.

The view of *actual infinity* regards itself as the *real objective infinite world*, so it must be the opposite of dialectical materialism. Therefore, *the actual infinity* is essentially a finite view, which abandons the difference between the objective world and the subjective world, and denies the unity of opposites between the subjective world and the objective world—*the contradiction between finity and infinity*, thus negatived the continuous progress of the subjective world.

2.5 Infinity Is an Objective Being

The contradiction between the subjective world and the objective material world is *a contradiction between finity and infinity*. The problem of the relationship between Thinking and Existence is also a problem of finity and infinity. *The contradiction between finity and infinity* is the concrete embodiment of the relationship between subject and object.

Infinity is an objective being. Time and space are infinite objects of reality that we can not escape. The being of limited human determines that this is a potential infinite being, a never-ending process. Plato and Aristotle did not deny the being of infinity, but Aristotle thought that infinity could not be realized realistically for finite human consciousness.

The subjective world is a finite world, and *finity* is a natural attribute of the subjective world. The objective material world is an infinite world. *The contradiction between finity and infinity* is a contradiction that human beings face when they understand the objective material world. It is that the human subjective world (finite) at this moment cannot fully understand and grasp the endless objective material world (infinite).

For more details, see Section 6 of my article *Philosophical Infinity and Mathematical Infinity* (see Zhang H. & Zhuang Y., 2019).

2.6 Bad Infinity Is a Fundamental Problem

Neither the standard nor the nonstandard real number model can get rid of the *bad infinity problem* (the problem of *the contradiction between finity and infinity*), that is to say, *bad infinity* is a fundamental problem.

The theory of *actual infinity* has never been separated from *bad infinity*. In fact, in his set theory, Mr. Cantor has been using *bad infinity*. For example, the generating process of transfinite ordinal number ω, starting from any ordinary natural number *n*, cannot reach ω, and this is a *bad infinity*. That is to say,

mathematicians who insist on the concept of *actual infinity* simply cannot leave the *bad infinite process.* It can be said that without the *bad infinity*, there would be no set theory of Cantor. According to the second principle of generation, the new number is ω, which is the essence of *bad infinity*. On the other side of the infinite process, ω can not embody the intrinsic nature of the set of natural numbers N.

The *bad infinity* always exists. In the nonstandard model, from the finite natural number *n* to the infinite number *b* is inaccessible and discontinuous, because *b* is larger than all finite natural numbers, so this process is a *bad infinite process*. Even if it is thought that from the finite natural number *n* to *b* is reachable, there is a *super finite chain* between them, but what is it after *b*? This is another *bad infinite process*. This is a disguised return to the motion of the standard model at infinity. Therefore, for nonstandard models, we can call them as standard models in infinite distance and infinitesimal place. However, in the standard models, we regard *very large or large natural numbers* or *sufficiently large numbers* as *b*. This is why the equivalence of standard analysis and non-standard analysis has been proved.

Therefore, Hegel firmly opposes such *actual infinity*: the finite is steadfast on this, the infinite is abiding on yonder side and infinite process has nothing to do with it. *b* and *ω* are the typical *actual infinity*, the product of two different worlds: finite real numbers and infinity, we can regard ω as the starting point of a new straight line.

What is *the completed infinity?* It never meant to complete an infinite process. In Cantor's view, this is a form in which an infinite object is represented as a number, such as the amount of natural numbers set N = {1, 2, 3……} is represented by ω. This is the thinking of Cantor, A.Robinson and Leibniz, but neither of them has left the *bad infinity*, and it is impossible to leave the bad infinity. It is their dream, their good will, or their wishful thinking, to use a number to represent the *"size, how much"* of an infinite object. But there is always a greater infinity, and the process is endless, the process itself is a new *bad infinity*. This *completed infinity* (that is, *actual infinity*) will only give us a moment, temporary peace, just a moment of peace, because there is always a bigger infinity waiting for us. The mistake of this concept lies in its emphasis on stillness and negation of motion, but *the contradiction between finity and infinity* exists forever, static is relative, and motion is absolute and eternal.

2.7 The Cantor-Hilbert Diagonal Argument and the One-to-One Correspondence Principle Are Essentially Finite Methods

There is a core and basic concept in Cantor's theory of transfinite numbers, which is *the principle of one-to-one correspondence*. Cantor's division of infinity is based on *the one-to-one correspondence principle.* But *the one-to-one correspondence principle* itself is a concept that needs to be clearly explained in the case of *the problem of finity and infinity*. How can we know that what is right in a finite category is also true in an infinite category? *The principle of one-to-one correspondence* is bound to encounter the problem of *bad infinity* in the face of infinity, and the division of infinity based on this doubtful concept becomes absurd in itself. *The principle of one-to-one correspondence* is the

foundation of Cantor's thought of *actual infinity*, so *the principle of one-to-one correspondence* based on finity cannot be used as a basis for evaluating the equivalence between *mathematical infinities*.

The one-to-one correspondence principle or *one-to-one mapping* essentially comes from and belongs to the category of finity, and when it is pushed to an infinite category, it can only be an axiomatic provision or assumption, rather than a *necessity* which can in fact be proved or inferred, for we cannot exhaust an infinite object. Thus, the concept of *equivalence of infinities*, which is based on *the one-to-one correspondence principle*, has been the center of the problem and the most doubtful thing.

Another important difference between finite and infinite is that finite is a *closed interval* with beginning and end, and infinity is *open interval* with no end. This profoundly reflects the great difference between finite category and infinite category.

In his book *The Logical Foundation of Mathematics and Infinity*, Mr. Zhu Wujia also questioned the application of *the one-to-one correspondence principle* to infinite sets. He pointed out, "*The one-to-one correspondence principle* used in infinite sets is also an enumeration procedure, but before enumerating the enumeration procedure, it will always be a kind of present progressive (going), so that it faces and refers to the potential infinity" (Zhu, 2008, p. 211). In the last part of chapter 7 of this work, this idea is further strengthened. He said, "In fact, in the traditional set theory, the establishment of the concept of *Power* is entirely determined by *the one-to-one correspondence principle*. In addition to giving a corresponding rule (or corresponding function), the only thing left for the use of *the one-to-one correspondence principle* is any recursive enumeration of elements in a set. But the elements of any recursive enumeration set can only be a potential infinite process at most, so it can only be applied to the potential infinite elastic set at most. There is no basis for Cantor's arbitrary application of *the one-to-one correspondence principle* based on arbitrary recursive enumerations to *actual infinite* rigid sets, especially for arbitrary applications to uncountable sets, because even in the traditional concept of set theory, it is also recognized that any recursive enumeration is at most up to countable infinity. That being the case, how can we determine the *power* of an uncountable infinite set according to *the one-to-one correspondence principle* based on arbitrary recursion? So it is very limited to use *the one-to-one correspondence principle* to determine the *power* of all kinds of *actual infinite* rigid sets" (Zhu, 2008, p. 296).

On the effectiveness of Cantor-Hilbert Diagonal Argument based on the reasoning of *actual infinite* thought, Mr. Zhu Wujia also gave a clear-cut question (Zhu, 2010). *The principle of one-to-one correspondence* and *Cantor-Hilbert Diagonal Method* are essentially *potentially infinite*, procedural methods, and are impossible completed methods, which cannot be used to prove the integrity of *actual infinity (global infinity)*. When we deal with *the Hilbert Hotel Problem*, if we are based on the same idea of *actual infinity*, we will get *the Infinite Exchange Paradox*. The method of taking the natural number set N as a whole is essentially a finiteness method, as defined in the power set of N, this is a finite, holistic approach. Two different infinities, we will not be able to distinguish them unless we know in advance the *relationship* between them (that is, the intrinsic qualitative connection); for

example, there exists a power-set relation between the natural number set N and its power set P (N), it is based on this relation that we can distinguish them and compare their size. We can only compare the size of different infinities in the sense of *real infinity*, that is to say, compare the size of different infinities in the connection between inner qualities of these infinities. In other words, the different infinities with internal connection can compare the size, and the different infinities without internal relation cannot compare the size. It is on this basis that we think that *the Axiom of Choice* is correct.

2.8 Philosophical Explanation of the Second Mathematical Crisis

The solution of the Second Mathematical Crisis is that the mathematicians inadvertently followed the thought of dialectics and gave the scientific definition of limit. However, Hegel analyzes the metaphysical confusion existing in mathematics based on dialectical thought, and makes a correct philosophical exposition on the concept of *limit*, and reveals the thought of bad infinity and real infinity in mathematics from the law of the mutual change of quality and quantity, so that we can thoroughly understand the essence of *limit*.

Hegel first described the mathematical confusion caused by metaphysics, “The mathematical infinite has a twofold interest. On the one hand its introduction into mathematics has led to an expansion of the science and to important results; but on the other hand it is remarkable that mathematics has not yet succeded in justifying its use of this infinite by the Notion (Notion taken in its proper meaning)” (Hegel, 1969, p. 240).

Hegel affirmed that infinitesimal is a process, a variable. Hegel and Kant object to regard infinity as a pure quantum (bounded quantity), as a maximum, as the completed quantity of a certain unit, that is, as an *actual infinity*. The essence of *actual infinity* is to treat infinity as a quantum, a limit, not a variable. Hegel pointed out, “Now since the infinitely great or small is that which cannot be increased or diminished, it is in fact no longer a quantum as such” (Hegel, 1969, p. 243), and “Kant objects to infinite wholes being regarded as a maximum, as a completed amount of a given unit.” (Hegel, 1969, p. 243), that is, Hegel and Kant think infinity and infinitesimal are one variable, which is completely consistent with *the limit theory* of modern mathematics. Infinity as a process is a variable, and the essence of infinity is movement and change. Therefore, Hegel’s philosophy is essentially a philosophy of motion, which fully reflects the bad infinity of the infinite. Infinity and infinitesimal are both one variable, one infinite progress, and Hegel explains: “The infinite quantum as infinitely great or infinitely small is itself implicitly the infinite progress; as great or small it is a quantum and at the same time it is the non-being of quantum.” (Hegel, 1969, p. 238), “The continuity of quantum with its other produces the conjunction of both in the expression of an infinitely great or infinitely small. Since both still bear the character if quantum they remain alterable, and the absolute determinateness which would be a being-for-self is, therefore, not attained.” (Hegel, 1969, p. 227). The infinitesimal quantity in the limit is a variable, neither a very small quantity nor an arbitrary small quantity, but a variable with zero as the limit, so it is only a disappearing quantity, which is completely consistent with the limit theory of modern mathematics. The reason why the dialectical essence of infinitesimal concept is not clear for a

long time lies in the inability to correctly understand the qualitative *moment* (or the qualitative stipulation) appearing in the infinite quantity. Hegel's amazing feat on infinity is the outstanding embodiment of his dialectic thought.

Hegel holds that *infinite quantum* includes not only the quantitative determinateness (that is, infinite progress, *bad infinity*), but also the qualitative determinateness (that is, *real infinity*). Real infinity is a qualitative stipulation; the reason why dx/dy belongs to real infinity is that it is no longer purely quantitative. Hegel pointed out, "It is a quantitative determinateness in qualitative form; its infinity consists in its being a qualitative determinateness." (Hegel, 1969, p. 245), the mathematical infinity here is *real infinity*. The so-called finite form of expression (fraction itself) is precisely the form of expression of *real infinity*, while the infinite of series is truly the form of finite expression, because it obviously belongs to the bad infinity such as infinite progress. "……in general, with the *functions of variable magnitudes.* This infinite is the *genuine mathematical* qualitative infinite which Spinoza also had in mind." (Hegel, 1969, p. 251), so dx/dy is a real infinity. Bad Infinity is only a kind of quantitative determinateness, which shows the contradictory movement and infinite progress between pure quantum and quantum. The real infinity is the quantitative determinateness of quality, which forms the contradictory movement of quantity and quality; because this contradiction is intrinsic and inevitable, it achieves the real unity of finite and infinite (He, 1983). So Hegel concludes: "The infinite quantum as the unity of both moments, of the quantitative and qualitative determinateness, is in the first instance a ratio." (Hegel, 1969, p. 314) This ratio is real infinity, and the limit concept is the *infinite quantum* here.

Infinitesimal is a disappearing quantity (variable), neither being (any quantity) nor nothing (pure zero), but the unity of being and nothing. Dx, dy, as an infinitesimal quantity, are no longer quantum; their meaning is only in relation, only means the *moment*, that is, the determinations of the differential co-efficient dx/dy; So they are both zero and not zero, not zero in the process of change, but the trend is zero. So Hegel stated, "Dx, dy, are no longer quanta, nor are they supposed to signify quanta; it is solely in their relation to each other that they have any meaning, a *meaning merely* as *moments.* They are no longer *something* (something taken as a quantum), not finite differences; but neither are they *nothing*; not empty nullities. Apart from their relation they are pure nullities, but they are intended to be taken only as moments of the relation, as determinations of the differential co-efficient dx/dy." (Hegel, 1969, p. 253)

Hegel holds that mathematical infinity is *real infinity* (referring to limit or dx/dy), and severely criticizes metaphysical infinity (refers to *bad infinity*). Hegel believes that the usual metaphysical infinity is *bad infinity*, and he points out: "while the ordinary metaphysical—by which is understood the abstract, spurious infinite" (Hegel, 1969, p. 249). Hegel puts forward that the infinite in the function of variables is a kind of real infinity of quality, while the bad infinity is pure negation, lacking the meaning of positive quality. Hegel's idea of real infinity helps to understand the dialectical relationship between result and process, infinite and finite, static and motion, quantitative change and qualitative

change in limit theory. In essence, the limit theory of modern calculus embodies the interrelation and transformation between finite and infinite, the unity of infinite quantity and infinite quality, the unity of quantity and variable, and the dialectical unity of real infinity and bad infinity. With regard to this contradiction in mathematics, Hegel made a clear distinction between mathematical infinity (real infinity) and metaphysical infinity (bad infinity), he pointed out that: "But in a philosophical respect the mathematical infinite is important because underlying it, in fact, is the notion of the genuine infinite and it is far superior to the ordinary so-called *metaphysical infinite* on which are based the objections to the mathematical infinite. Often, the science of mathematics can only defend itself against these objections by denying the competence of metaphysics, asserting that it has nothing to do with that science and does not have to trouble itself about metaphysical concepts so long as it operates consistently within its own sphere. Mathematics has to consider not what is true in itself but what is true in its own domain. Metaphysics, though disagreeing with the use of the mathematical infinite, cannot deny or invalidate the brilliant results obtained from it, and mathematics cannot reach clearness about the metaphysics of its own concept or, therefore, about the derivation of the modes of procedure necessitated by the use of the infinite." (Hegel, 1969, p. 241). Engels, in *Dialectics of Nature*, also expounds the profound role of mathematical infinity (that is, real infinity), "The differential calculus for the first time makes it possible for natural science to represent mathematically *processes* and not only *states:* motion." (Engels, 1964, p. 277).

Hegel thought that the infinite of mathematics is important in philosophy, because it is based on the foundation of *real infinity*, much better than metaphysical infinity. The infinity of mathematics and the infinity of metaphysics are important problems in the theory of infinity. The true mathematical infinity is *real infinity*, and the infinite of metaphysics is *bad infinity*. The infinite of metaphysics insists on the external negativity (*bad infinity*), and the boundary is never sublated; but *real infinity* insists on the intrinsic negation, and this kind of negation is the return to itself, the unity of the quantity and the quality of itself. The limit and the disappearance of the quantitative relationship do not mean that the disappearance of the qualitative provisions. Hegel emphasized the ratio of quality, because it is the basis for the conversion of a finite quantity to an infinite quantity. Therefore, it is very important for the healthy development of mathematical science to take seriously *the view of actual infinity* and *the view of potential infinity* in mathematics.

However, as Hegel said, "the only lesson that man has learned from history is that man has not learned any lessons from history", and man is constantly repeating the same mistakes. After Cantor invented *infinite number* (base number, ordinal number) more than 100 years ago, people's understanding of infinity once again fell into the metaphysics of idealism, which led to the outbreak of the Third Mathematical Crisis.

2.9 Size Comparison of Infinity

On the size problem between infinity Hegel gave a clear argument. Hegel holds that infinity is comparable and can be compared only in the sense of *real infinity*, that is, in the sense of the qualitative determinateness of infinity. What is in *ratio* is no longer quantitative, but qualitative, Hegel pointed out that: "on the other hand the qualitative is what it is only in its distinction from an other. The said infinite magnitudes, therefore, are not merely comparable, but they exist only as moments of comparison, i.e., of the ratio." (Hegel, 1969, p. 255). The two infinities in the *ratio* are two infinite advances, and their rate of change is a *real infinity*, reflecting the size of a qualitative determinateness.

Two different infinities, we will not be able to distinguish them unless we know in advance the *relationship* between them (that is, the intrinsic qualitative connection); for example, there exists a power-set relation between the natural number set N and its power set P (N), it is based on this relation that we can distinguish them and compare their size. We can only compare the size of different infinities in the sense of *real infinity*, that is to say, compare the size of different infinities in the connection between inner qualities of these infinities. In other words, the different infinities with internal connection can compare the size, and the different infinities without internal relation cannot compare the size. It is on this basis that we think that *the Axiom of Choice* is correct.

2.10 Attach Great Importance to the Infinite Thought of Wittgenstein, the Master of Philosophy

Wittgenstein is one of the most influential philosophers in the 20th century, and especially his thought of philosophy of mathematics attracted a long debate. His infinite thought mainly is: he opposes the *actual infinity*, the objective existence of infinity, thinks that the infinity is a kind of infinite possibility expressed by the law, but not the reality; he is opposed to the one-to-one correspondence of an infinite set to his own subset and to the use of *Cantor Diagonal Argument*, so he is a typical theorist of *potential infinity*.

He denied the infinite reality. He believed that infinite reality could not be proved, and that symbols could not express infinite reality. As he said in *Research in Basic Mathematics*, "it says that *actual infinity* cannot be grasped by mathematical symbolic systems at all, so it can only be described and not shown. This description may have been grasped in a way similar to the following: a large number of things that cannot be held in hand by people are lifted by packing them in boxes." (Wittgenstein, 2013, p. 210)

He opposes the use of *Contor Diagonal Method*, arguing that the infinite cannot be exhausted by finity. "For we have a legitimate feeling that where we can talk about the last thing, there can be no '*no last thing at all*'." (Wittgenstein, 2013, p. 207); "Please don't forget: mathematicians' thoughts on infinity are, after all, finite ones. Here's what I want to say: they all have an end." (Wittgenstein, 2013, p. 228).

He holds that infinity and finity are completely different categories, and infinity is an inherent stipulation. He thinks that infinity is not a number, it is not the size of a quantity competing with a finity, but an inherent stipulation. He pointed out that, "*Infinite sets* and *Finite sets* are two different logical categories. Something that can be meaningfully expressed to one category cannot be

meaningfully expressed to another category." (Wittgenstein, 2013, p. 206). Taking the number π as an example, the number π expresses an infinite law which is accompanied by the actual observation, that is, the number π is a rule. In essence, this is Hegel's thought of *real infinity*.

He thinks infinity is a possibility. In his opinion, the difference between finity and infinity is not a difference in quantity, but a logical difference; infinity is not a quantity or an extension; infinity is an infinite possibility expressed by law, and infinity itself is not comparable in size. So what he thinks of as infinite possibility is actually a variable, a process (*bad infinity*) rather than an outcome. Thus, Wittgenstein was again a theorist of *potential infinity*.

Therefore, Wittgenstein's infinite thought basically obeys Hegel's dialectical infinite thought. His only deficiency is his denial of infinite objectivity, and he was with the intuitionists but overtook them. However, his infinite thought failed to rise to the level of Hegel's dialectical infinite view, and failed to grasp the philosophical significance of infinity as *free self infinity—real infinity*.

2.11 Adhere to the Infinite View of Dialectical Materialism

Redefine, recognize the *actual infinity* and the *potential infinity*, fully absorb Hegel's thought of dialectical infinity, return to the scientific concept of infinity, that is, infinity is an objective Being, infinity can be known, but the infinite process cannot be completed. And this is the infinite view of dialectical materialism.

To sum up, we comprehensively introduce Hegel's dialectical infinity, and also introduce Engels's critical inheritance and development of Hegel's infinite view, it also introduces the differences between the four kinds of infinite views, and analyzes the errors of *actual infinity*, so now we can naturally sum up *the infinite view of dialectical materialism*: infinite objective exists, infinite can be known, but *infinite process* can not be completed. Concretely speaking, any infinity is the dialectical unity of *bad infinity* and *real infinity*. It is an objective existence, and infinity itself contains *the contradiction between finity and infinity*, so the objective existence of infinity does not mean that the infinite process can end, complete. *Real infinity* is the inherent qualitative stipulation of infinite things, that is, the inner connection, law and truth, while *bad infinity* is infinite progress, no terminating repetition and alternating, it deeply embodies *the contradiction between finity and infinity*. *Real infinity* can be recognized and completed, while *bad infinity* cannot be recognized and can not be completed; *bad infinity* (infinite process) is the concrete manifestation of *the contradiction between finity and infinity* rather than the solution of such contradiction, and this determines that *the contradiction between finity and infinity* will never die out. *Real infinity* represents infinite quality (essence), while *bad infinity* represents infinite quantity (movement and change). *Real infinity* is inseparable from *bad infinity*, *bad infinity* is the carrier of *real infinity*, and *real infinity* is the goal and direction of *bad infinity*. The *real infinity* is present, concrete, positive, rational, completed infinity, is *Being-for-self* and rational *Being*, is the completed quality; and the *bad infinity* is possible, abstract, negative, uncompleted infinite, is *Being-in-itself* and *intellectual Being*.

Infinity exists and cannot be traversed, completed; infinity is a black hole, but it can be sublimated and transcended. Process (*bad infinity*) forever, law (*real infinity*) eternal; process is a variable, law is a constant; Infinity cannot be crossed, but can be transcended, the result of transcendence is a real infinity. The existence of infinity and the imcompletion of process are two completely different concepts, they are two aspects of contradiction and cannot be replaced by each other; it is precisely because of their existence that there is the existence of *the contradiction between finity and infinity*. Infinity, like a black hole, can come in without going out, endless, never over. The infinite view of dialectical materialism insists on the inextinguishment of this kind of contradiction, and holds that infinity exists objectively and can be recognized, but the infinite process cannot be completed, that is, the infinite contradiction is eternal; On the other hand, the view of *actual infinity* regards the objective existence of infinity as the completion of the infinite process, replacing the subjective with the objective, replacing the *bad infinity* with the *real infinity*, abandoning *the contradiction between finity and infinity* completely, and thinking that the contradiction can be ended and solved. Therefore, this thought followed Kant's transcendental, subjective, metaphysical infinite thought, not Hegel's dialectical infinite thought.

3. Root Causes and Solutions of the Third Mathematical Crisis

The above-mentioned we discuss the insurmountable inherent contradiction in the view of actual infinity, which always treats infinite objects in a finite and mechanical way, thus bringing one bigger, worse contradiction after another. We believe that the essence of the Third Mathematical Crisis is the crisis caused by the view of *actual infinity*—exhausted an infinite (*exhausted an inexhaustible thing*). The Max Ordinal Paradox, the Max Cardinality Paradox and Russell's Paradox all embody the fundamental error of *actual infinity*.

Infinity, as a being, can not be defined by a restrictive concept, such as the concept of *fixed infinity*, which can not describe the real infinite object; once a boundary is given, this infinity becomes a finite. Limited infinity is not really infinite, but a finite. *Actual infinity* is such a limited infinity, and *finished infinity* is a limited infinity, such as the maximum cardinal number, maximum ordinal number, the Principle of Comprehension.

We know that the Third Mathematical Crisis is caused by paradoxes in set theory. Among them, Russell's Paradox is the core. After that, mathematicians gave axiomatic solutions (such as ZFC set theory), but only form to solve the crisis, it is not known whether there is a definitive solution to the crisis. This is because the mathematical world has not found the crux of the problem, did not realize that *the thought of actual infinity* is the culprit of the crisis, it led to circular judgment.

Because the non-contradiction of ZFC system itself has not been proved so far, there is no guarantee that there will be no paradox in this system in the future. While those emerged paradoxes of set theory can be ruled out in ZFC system, no other contradiction has occurred, and ZFC system has been applied to today. But Poincaré pointed out: we set up a fence to surround the sheep from wolves, but it is likely

that a wolf was surrounded by the fence. Because the ZFC system can not guarantee that there will be no paradox in the future, the Third Mathematical Crisis has not been solved completely in this sense.

Mr. Du Guoping makes a systematic analysis of the causes of Russell's Paradox in *Research Progress of Russell's Paradox* (see Du Guoping, 2012). He thinks that the cause of the paradox lies not in the logical system, but in the Principle of Comprehension or the basic definition of set theory. It is pointed out in the article *Set Theory- Universal Logic Paradox* (see Du Guogong, 2009) that the Principle of Comprehension will lead to paradoxes in the finite logic, countable infinite logic and uncountable infinite logic systems.

However, the root of the problem lies in the Principle of Comprehension, because the philosophical thought on which the Principle of Comprehension (as the basic principle of constructing the set) depends is the thought of *actual infinity*. The judgment of *all objects* is the judgment of *finished infinity*, which is a kind of actual infinite thought.

What is the Principle of Comprehension? Principle of Comprehension is the basic principle of classical set theory, which refers to an important stipulation or axiom used to construct set in classical set theory. The content is unconditional recognition of any nature P (or property P), and one can bring together all the objects that satisfy the P of that nature and form a set only by bringing together these objects with P nature. The symbol is G={x |P(x)}, where the x on the left of "|" represents any element of the set G, and the P(x) on the right of "|" means that the element x has property P, and { } means that all x with property P are brought together to form a set. Therefore, another expression of the Principle of Comprehension is $\forall x(x \in A \leftrightarrow P(x))$. That is, the elements of the set A must have property P, on the contrary, all objects with property P must be the elements of the set A. Therefore, the Principle of Comprehension is a set existence axiom.

Under the Principle of Comprehension, there is no limit to the object domain. It is this unlimited *"all objects"* (actual infinity) that leads to paradox. There are two understandings of this kind of *"all"*, one representing *"existing"* (actually a potential infinity), the other representing *"existing"* and *"coming"* (which is practically an actual infinity). Obviously, the emergence of paradox is due to the latter understanding. The previous understanding that the object of judgment is oriented to history rather than to the future is therefore a potential infinity; the latter understanding is both historical and future-oriented, which is clearly a judgment of actual infinity. This involves determining the criteria for a judgment object, whether it is a judgment of existence, or a judgment of an upcoming future.

According to the fact that human cognition of the objective world, time unidirectionality and directionality of judgment show that our human thinking can only judge the exact existing objects, and this is determined by the hierarchy of knowledge, the law of historical development, and the law that the world is hierarchical. The law of cognition, the unidirectional nature of time and the directionality of judgment are a potential infinite thought. The application of Axiom of Separation in the axiom system ZFC, the essence is to limit the object of our judgment to the existing range, that is, to limit the definition of set to the range determined jointly by known object (given set) and given property. For

example, it does not allow the collection of all sets to exist, which is an embodiment of potential infinity. As a result, the ZFC axiom system can fundamentally eliminate the emergence of paradox. As can be said, ZFC axiomatic set theory is a potential infinite set theory, which makes up for the deficiency of naive set theory.

On the other hand, if we adhere to the latter standard of judgment, that is, *future* is also included in our vision of judgment, which will inevitably lead to circular judgment, that is, *present* judgment is the judgment of *past and future*, which leads to the emergence of Russell's Paradox. This is not only an *actual infinite thought*, but also a violation of the principle of time unidirectional, which will inevitably bring confusion to our understanding.

Next, we analyze the Maximum Ordinal Paradox in detail.

The Maximum Ordinal Paradox, also known as the Burali-Forti's Paradox, it is the first paradox in the history of set theory. Let W be a set of all ordinal numbers, that is, W={1/2,...,ω,…}. N={1,2,3,..., n,…} is a set of natural numbers, the ordinal number of N is ω, ω greater than any natural number, and it is very important that the ω does not belong to the N. We can see that W form a well-ordered set in order of ordinal number, according to the definition of well-ordered set, so the W itself also corresponds to an ordinal number Ω, and it is Ω the largest ordinal number, which is larger than any ordinal number in the W, but by W definition, we know that Ω also appear in the W, thus there will be Ω>Ω, and this is contradictory. At the Baromol Society of Mathematical Sciences, Italy, on March 28, 1897, C.Burali-Forti read out a paper, put forward the above paradox, opened the prelude of the third crisis of mathematical foundation. Actually, the German mathematician Cantor discovered this paradox two years ago, but it was not made public.

We know that the general process of ordinal definition is a potential infinite process, and all new ordinal numbers do not belong to the forefront, such as ω do not belong to the N. However, people want to complete and end this infinite process of defining ordinal numbers once and for all, —this is an *actual infinite thought* that defines all ordinal numbers, therefore, a well-ordered set composed of all ordinal numbers is produced, and this set also corresponds to an ordinal number, which brings a question: whether this ordinal number belongs to this ordinal set, which leads to contradiction. Thus, this infinite process can not end. Because infinity is movement and change, there is no limit, there is no end, this is the *potential infinite* (bad infinite) thought. Because ordinal numbers are constantly growing, that is, we can not define all ordinal numbers, the so-called set of all ordinal numbers becomes castles in the air, so it is impossible to exist.

Similarly, on the Russell's Paradox, it should be said that the structure of the set must conform to the *potential infinite principle*, that is, the Principle of Comprehension must follow the potential infinite principle. Construction is the judgment of existence (that is, the judgment of history), is oriented to history, not to the future; by adhering to this principle, we can clearly solve the Russell's Paradox.

With regard to *the problem of finite and infinite*, we know that there are internal irreconcilable contradictions in the mathematical basis, such as: the line segment is composed of dots, while the

"dot" has no measure, while the "*line segment*" has a measure; those dots without measure constitutes a measured line segment, which is obviously contradictory and absurd. To some extent, dots are imposed on a line segment by us humans. Because of this inherent contradiction in the mathematical foundation, it leads to many strange and absurd results in mathematics, which affects the image of mathematical certainty. The root of this problem lies in the lack of a dialectical understanding of the contradiction of finite and infinite. We have two acceptable understandings of the relationship between dots and segments: One is to think that *a line segment is made up of dots*, but it must be a dot has an *infinite small* measure (it is a variable), that is, a dot has a measure that is neither zero nor zero. Another idea is to deny that a segment consists of dots, that is, the line segment is not composed of dots, the line segment itself is a measured, a basic mathematical abstract object, the dot does not belong to the line segment, the dot is imposed on the line segment, and the dot is only a descriptive tool of measure and position. Both explanations can solve the problem of continuous and discrete contradiction well.

Therefore, correctly understanding and grasping *the contradiction between finity and infinity* and insisting on the dialectical understanding of infinity are the essence of solving the mathematical crisis thoroughly in our mathematical and philosophical circles, and also have great guiding significance for perfecting the mathematical foundation building.

In summary, we have carried on the comprehensive analysis to the infinite thought, and concluded that there is insurmountable internal contradiction in the view of *actual infinity* which can be completed by *infinite process*. At the level of philosophy and mathematics, Hegel puts forward the concept of *real infinity* (Marx and Engels further clarified and developed this thought) in accordance with dialectical materialism, which embodies the essence and relation of infinite object. This is something that the thought of *actual infinity* (this lower-level, metaphysical, subjective idealistic view of infinity) cannot be compared. We believe that adhering to *the infinite view of dialectical materialism* will bring a brighter future to the study of basic mathematics.

References

Aristotle. (2016a). *Physics*. Published by The Commercial Press (China).

Aristotle. (2016b). *Metaphysics*. Published by The Commercial Press (China).

Du, G. P. (2012). *Research Progress of Russell Paradox*. *Journal of Hubei University (Philosophy and Social Science) (China)*, *39*(5).

Du, G. P., Wang, H. G., Li, N., & Zhu, W. J. (2009). Set Theory-Universal Logic Paradox. *Journal of Beijing University of Aeronautics and Astronautics (China)*, *35*(3).

Engels, F. (1947). *Anti-Dühring* (Emile Burns, Trans.). Published by LeoPard Books India.

Engels, F. (1964). *Dialectics of Nature* (Clemens Dutt, Trans.). Published by Wellred 2012.

He, J. N. (1983). *Hegel's Idea of Infinity in Mathematics and Its Significance*. Foreign Philosophy (part 4) (China).

Hegel, G. W. F. (1969). *Hegel's Science of Logic* (A. V. Miller, Trans.). Published by Humanity Books.

Hegel, G. W. F. (2015). *The Logic of Hegel* (William Wallace, M.A., Trans.). Published by Forgotten Books.

Hume, D. (2015). *A Treatise of Human Nature*. Published by The Commercial Press (China).

Ling, E. (1992). The Three Mathematical Crises Caused by Bad Infinity. *Journal of East China Jiaotong University (China)*, *9*(1).

Marx, K. (1975). *Mathematical Manuscripts*. Published by People's Publishing House (China).

Poincaré, H. (2008). *The Foundations of Science*. Published by The Commercial Press (China).

Poincaré, H. (2015). *The Final Meditation*. Published by The Commercial Press (China).

Robinson, A. (1980). *Non-standard Analysis*. Published by Science Press (China).

Wen, C. R. (2003). Hegel's Thought about Quantity and Mathematical Infinity. *Journal of Anhui University (China)*, *2003*(3).

Wittgenstein, L. (2013). *Research in Basic Mathematics*. Published by The Commercial Press (China).

Xu, L. Z. (Hsu, L. C.). (2000). *Topics in Mathematical Methodology* (3rd ed.). Published by Huazhong University of Science and Technology Press (China).

Zhang, H., & Zhuang, Y. (2019). PHILOSOPHICAL INFINITY AND MATHEMATICAL INFINITY. *Philosophy of Mathematics Education Journal*, *2019*(35).

Zhu W. J. (2010). Analysis and Research of Cantor-Hilbert Diagonal Proof Argument. *Journal of Nanjing Xiaozhuang University (China), 2010*(3).

Zhu, W. J. (1987). *Foundations of Geometry and Mathematics*. Published by Liaoning Education Press (China).

Zhu, W. J. (2008). *The Logical Foundation of Mathematics and Infinity*. Published by Dalian University of Technology Press (China).