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Original Paper

Turing Machine Halting Problem, Russell's Paradox and Gödel Incompleteness Theorem

Hong Zhang[1]

[1] Bank of China, China

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Abstract

The Turing Machine Halting Problem is a major problem in computer theory, Russell's Paradox is the root of the Third Mathematical Crisis, and the ***Gödel Incompleteness Theorem*** *is a major discovery in modern logic. The three have had a profound impact on the development of science and have attracted the attention of scientific and philosophical circles. However, since the* ***Gödel Incompleteness Theorem*** *was put forward, the scientific and philosophical significance of its proof has been questioned; in particular, Wittgenstein regards it as a certain logical paradox, and Russell's Paradox has not yet been settled. This paper makes a detailed analysis of the three based on the view of dialectical infinity. The author notes that the Principle of Comprehension based on the view of actual infinity is the root of Russell's Paradox. The Turing Machine Halting Problem shows that it is impossible to make an actual-infinite ultimate judgment of the constantly generated infinite world, but the philosophical significance of the* ***Gödel Incompleteness Theorem*** *is that our understanding of the world is essentially potentially infinite. At the end of the article, the author raises several questions about the proof of the* ***Gödel Incompleteness Theorem****,* ***finds out the specific paradox form in the proof,*** *points out the high consistency of its proof method and Russell's Paradox, which strongly supports Wittgenstein's view. The author points out that the philosophical basis of the proof of the* ***Gödel Incompleteness Theorem is the*** *idea of actual infinity, the proof of the theorem is based on a logically invalid circular formula, the contradiction of the proof originates from the* ***Gödel*** *formula itself, and cannot be attributed to the incompleteness of the system, so the proof is wrong. Therefore, the conclusion of this paper is that the world is constantly developing and changing, and our human understanding of the world is essentially a potential infinite, that is, the world is Aristotelian, not Platonic.*

Keywords

the Turing Machine Halting Problem, Russell's Paradox, the Gödel Incompleteness Theorem, the Principle of Comprehension, the Cantor Diagonal Argument

In 1901, Russell discovered Russell's Paradox, which brought a third crisis to the development of mathematics. The solution of this paradox has not been concluded. Soon after, in 1931, the logician **Gödel** proved the famous incompleteness theorem, which made an epoch-making change in the study of the basis of mathematics. However, this theory has been strongly criticized by the philosophical master Wittgenstein. Although the circles of mathematics and logic tend to fully accept this theory, the debate between the two ideas has not been resolved. The Turing Machine Halting Problem is a major problem of computer theory, and its far-reaching influence has important philosophical significance. There are two ways to prove the Turing Machine Halting Problem (i.e., the **undecidability** of the Halting Problem): the Cantor Diagonal Argument and the Judgment Procedure Argument. These two methods of proof have been questioned by scientific and philosophical circles. The Cantor Diagonal Argument is questioned because the philosophy of this method is a view of *actual infinity*, which is impossible to complete in the view of *dialectical infinity* (Zhang H. & Zhuang Y., 2019).

1. The Infinite Exchange Paradox Reveals the Internal Irreconcilable Contradictions of the View of Actual Infinity

The so-called *Infinite Exchange Paradox* refers to the idea that we use the thought of *actual infinity*—that is, the idea that the infinite process can be accomplished, and we can transform the two equivalent infinities (with *one-to-one correspondence*) into mutually nonequivalent infinities. **This profoundly exposes the inherent defects of *the thought of actual infinity*; it moves the contradiction to infinity, but the contradiction never disappears.** In other words, the infinite process is impossible to complete, thus further supporting *the view of dialectical infinity*. This shows that the famous **Hilbert Hotel Problem** is not valid because the method of proof (the idea of *actual infinity*) is inappropriate.

The paradox is as follows:

We know one-to-one mapping $f: N \to W$, where N is the set of natural numbers and $W = \{a_1, a_2, a_3, \ldots\ldots\}$ is a countable infinite set. $\forall n \in N$, we have $f(n) = a_n \in W$.

Here are a series of successive transformations, and we think that the infinite process can be completed and ended. Let T represent the transformation, T (n) for the n-th transformation:

T(1): Exchanging the *elephants* of the natural numbers 1 and 2, so that 1 corresponds to a_2, and 2 corresponds to a_1;

After T (1) is completed, make the transformation T (2): that is, exchanging the elephants of 2 and 3, so that 2 corresponds to a_3, and 3 corresponds to a_1;……;

When T (n) is transformed, it occurs after the transformation T (n-1), that is , exchanging the elephants of n and n+1, so that n corresponds to a_{n+1} , and n+1 corresponds to a_1 . And so on, until infinity, we think this infinite process can be done.

At the end of this infinite continuous transformation, we will find the following contradiction: we can no longer find a natural number that corresponds to the element a_1 of W . This is the *Infinite Exchange Paradox.*

2. The Cantor Diagonal Argument Has Been Questioned

The above **Infinite Exchange Paradox** clearly shows that any infinite process cannot be completed, so the rationality of the **Cantor Diagonal Argument** is immediately put into doubt.

The great philosopher Wittgenstein firmly opposed the use of the ***Cantor Diagonal Argument***, arguing that infinity cannot be exhausted by finity. As he said in *Research in Basic Mathematics*, "For we have a legitimate feeling that where we can talk about the last thing, there can be no '*no last thing at all*'" (Wittgenstein, 2013, p. 207). In the second part of *On the Foundations of Mathematics*, he deeply criticizes and reflects on the *Cantor Diagonal Argument*: "I can indeed say here: there is always one in the sequence, and it is uncertain whether it is different from the diagonal sequence. One can say that they follow each other, tend to infinity, but the original sequence is always on top." (Wittgenstein, 2003, p. 82)

Mr. Zhu Wujia, a famous mathematical logician, also clearly questioned the ***Cantor Diagonal Argument*** (Zhu, 2010). The principle of *one-to-one correspondence* and the ***Cantor Diagonal Method*** are essentially potentially infinite, procedural methods and are impossible completed methods, which cannot be used to prove the integrity of *actual infinity* (global infinity).

The Judgment Procedure Argument is questioned because its idea of proof is consistent with Russell's Paradox, and the complete solution of Russell's Paradox has not been settled. Therefore, it is also natural for people to question the Judgment Procedure Argument.

3. Root Causes and Solutions of Russell's Paradox

The ancient paradox of ***Line Segments Are Composed of Dots*** and the ***Infinite Exchange Paradox*** profoundly reveal the *insurmountable inherent contradictions* in the view of *actual infinity*, which always treats infinite objects in a finite and mechanical way, thus bringing one bigger, worse contradiction after another. We believe that the essence of **the Third Mathematical Crisis** is the crisis caused by the view of *actual infinity* - exhausted an infinite (exhausted an inexhaustible thing). **The Max Ordinal Paradox**, **the Max Cardinality Paradox** and **Russell's Paradox** all embody the fundamental error of *actual infinity*.

Infinity, as a being, cannot be defined by a restrictive concept, such as the concept of fixed infinity, which cannot describe the real infinite object; once a boundary is given, this infinity becomes a finite. Limited infinity is not truly infinite but finite. *Actual infinity* is such a limited infinity, and finished infinity is a limited infinity, such as the maximum cardinal number, maximum ordinal number, and the Principle of Comprehension.

We know that **the Third Mathematical Crisis** is caused by paradoxes in set theory. Among them, Russell's Paradox is the core. After that, mathematicians gave axiomatic solutions (such as ZFC set theory), but only form to solve the crisis, it is not known whether there is a definitive solution to the crisis. This is because the mathematical world has not found the crux of the problem and **did not realize that the thought of *actual infinity* is the culprit of the crisis, which led to circular judgment.**

Because the ***noncontradiction*** of the ZFC system itself has not been proven so far, there is no guarantee that there will be no paradox in this system in the future. While those emerged paradoxes of set theory can be ruled out in the ZFC system, no other contradiction has occurred, and the ZFC system has been applied today. However, Poincaré pointed out that we set up a fence to surround the sheep from wolves, but it is likely that a wolf was surrounded by the fence. Because the ZFC system cannot guarantee that there will be no paradox in the future, **the Third Mathematical Crisis** has not been solved completely in this sense.

Mr. Du Guoping systematically analyzes the causes of Russell's Paradox in *Research Progress of Russell's Paradox* (Du, 2012). He thinks that the cause of the paradox lies not in the logical system but in the Principle of Comprehension or the basic definition of set theory. It is pointed out in the article *Set Theory-Universal Logic Paradox* (Du, 2009) that the Principle of Comprehension will lead to paradoxes in finite logic, countable infinite logic and uncountable infinite logic systems.

However, the root of the problem is precisely in the Principle of Comprehension, because the philosophical thought on which the Principle of Comprehension (as the basic principle of constructing the set) depends is the thought of *actual infinity*. The judgment of all objects is the judgment of finished infinity, which is a kind of *actual infinite thought*.

What is the **Principle of Comprehension?** The Principle of Comprehension is the basic principle of classical set theory, which refers to an important stipulation or axiom used to construct sets in classical set theory. The content is unconditional recognition of any nature P (or property P), and one can bring together all the objects that satisfy the P of that nature and form a set only by bringing together these objects with P nature. The symbol is G={x |P(x)}, where the x on the left of "|" represents any element of the set G, the P(x) on the right of "|" means that the element x has property P, and { } means that all x with property P are brought together to form a set.

Therefore, another expression of the Principle of Comprehension is $\forall x(x \in A \leftrightarrow P(x))$. That is, the elements of set A must have property P; in contrast, all objects with property P must be the elements of set A. Therefore, **the Principle of Comprehension** is an axiom about the existence of sets (in the axiom mode).

Under the Principle of Comprehension, there is no limit to the object domain. It is this unlimited ***all objects*** (*actual infinity*) that leads to paradox. There are two understandings of this kind of *all*, one representing *existing* (actually a *potential infinity*) and the other representing *existing* and *coming* (which is practically an *actual infinity*). Obviously, the emergence of paradox is due to the latter understanding. The previous understanding that the object of judgment is oriented to history rather than to the future is therefore a potential infinity; **the latter understanding is for both the history and the future, which is clearly a judgment of *actual infinity*.** This involves determining the criteria for a judgment object, whether it is a judgment of existence or a judgment of an upcoming future.

The so-called finished infinity (that is, infinite progress) is not a definite quantity, but a quantity of change. How can we judge it? Therefore, this judgment of all objects is impossible and thus invalid.

According to the fact that human cognition of the objective world, *unidirectionality* of time and *directionality* of judgment show that our human thinking can only judge the exact existing objects, this is determined by the hierarchy of knowledge, the law of historical development, and the law that the world is hierarchical. The law of cognition, the unidirectional nature of time and the directionality of judgment are a thought of *potential infinity*.

On the other hand, if we adhere to the latter standard of judgment, that is, the future is also included in our vision of judgment, it will inevitably lead to circular judgment, that is, **present judgment is the judgment of past and future, and the result of present judgment must belong to the category of future, which leads to circular judgment and the emergence of Russell's Paradox.** This is not only a thought of *actual infinity* but also a violation of the principle of time unidirectional, which will inevitably bring confusion to our understanding.

The ZFC axiomatic set theory is proposed to avoid Russell's Paradox. There are two axioms worth our attention: **the Axiom Schema of Separation and the Axiom of Regularity.**

The application of the Axiom Schema of Separation and Axiom of Regularity in the ZFC axiom system is a limitation on **the Principle of Comprehension. The essence is to limit the objects of our judgment to the existing range, that is, to limit the definition of set to the range determined jointly by known objects (given sets) and given property.** For example, **it does not allow the set of all sets to exist and the set sequence of infinite descending chains, which is exactly the embodiment of *potential infinity*.** The composition of all sets can be traced back to minimal elements (not sets in themselves). Therefore, the sets formed by the ZFC axiom system are the judgment of history, the judgment of existing objects, and even the existence of infinite sets, such as the set of natural numbers, are also the judgment of objects already existing. Therefore, its foundation is based on the solid *earth* (the *earth* composed of the real *minimal elements*), so it must fundamentally eliminate the emergence of paradox. ZFC axiomatic set theory is a set theory of *potential infinity*, which makes up for the deficiency of naive set theory. Let us specifically analyze the role of the two axioms.

3.1 Axiom Schema of Separation

The Axiom Schema of Separation is also called the Axiom Schema of Specification and the Axiom Schema of Restricted Comprehension. The implication is that, given any set and any proposition P(u), there is a subset of the original set that contains and only contains the elements that make P(u) valid. Its essence is a limitation on the Principle of Comprehension. It is specifically expressed as follows:

Make **P(u)** a formula, and for any set **A**, there is a set **Y={u∈A｜P(u)}**; the logical expression is $\forall A \exists Y \forall \mathrm{u}(u \in Y \Leftrightarrow u \in A \wedge P(u))$. It actually represents infinitely many axioms, and for each formula P, there is a corresponding axiom of separation.

Obviously, set Y is a restriction on the existence of set A, and set A is in turn a condition for the existence of set Y. This limitation is as follows: when $P(u) = u \notin u$, we must have $Y \notin Y$; otherwise, if $Y \in Y$, there is $Y \in A \text{and } Y \notin Y$, leading to contradiction. In this case, $Y \notin Y$ no longer leads to conflict, because it contains $Y \notin A$. Therefore, this axiom contains the following inference: for any set A, there is always a set Y, making $Y \notin A$. Therefore, we conclude that a set of all sets does not exist.

3.2 Axiom of Regularity (Axiom of Foundation)

The definition is that each nonempty set x always contains an element y, so x disintersects with y (or has no intersection). Its logical expression is $\forall x(x \neq \varnothing \Rightarrow \exists y(y \in x \wedge x \cap y = \varnothing))$.

Its direct inference is that any set x does not belong to itself and ensures that there is no infinite descent chain of sets. That is, for all sets, we constantly trace the elements of its elements, and always stop within finite steps. This also shows that the formation and generation of all sets are the judgment of the already existing objects. This is clearly a definition of a *potential-infinite* nature.

The Principle of Comprehension is to collect all the *objects* with property P(x) into a set without any restrictions, but it does not care whether these *objects* exist and what they are; **the Axiom of Regularity** tells us what these *objects* truly are, and **the Axiom Schema of Separation** ensures that the judgment of all these *objects* is feasible, thus pooling a group of existing objects with property P(x) into a set. **This ensures that the ZFC axiom system is a set theory based on the idea of *potential infinity*,** thus completely eliminating the paradox; instead, naive set theory based on the Principle of Comprehension is based on the idea of *actual infinity*, which leads to the emergence of Russell's Paradox. **Therefore, the essence of the Third Mathematical Crisis lies in that we adhere to the thought of *actual infinity*, which is the crisis caused by the thought of *actual infinity*.**

4. The Analysis of Philosophical Thought Behind the Turing Machine Halting Problem

First, let us introduce what **the Turing Machine Halting Problem** is. The Turing Machine Halting Problem is: given any Turing Machine M and the input alphabet ∑, and any string **w** $\in \sum^*$ ($\sum^*$ is a set of

strings on ∑), will it stop to run the input w on M? This problem is equivalent to the following judgment problem: whether there is a program P, for any input program w, can judge that w will end or enter an endless loop in a limited time.

We already know that the **Turing Machine Halting Problem** has been solved, that it is *undecidable*, and that we do not have an algorithm to solve it. In the proof, we assume that there is such a Turing machine H that can solve the Halting Problem, and we can always construct a new Turing machine D on the basis of H. This leads to a contradictory result that we find D is down on the input R (D) (*the string representation of Turing machine D*) if and only if D is not down on R (D).

In fact, the **Turing Machine Halting Problem** is essentially the same as **Russell's Paradox**, both appearing because of the idea of *actual infinity*. As Russell's Paradox, based on the idea of *actual infinity*, we believe that *all* sets that do not belong themselves can form a set T, and this *all* is a *completed all*, namely, an *actual infinity*. Thinkers of *actual infinity* argue that this newly defined set T also belongs to this *all*, leading to the paradox that T belongs to both itself and not himself. Of course, in the view of the thinkers of *dialectical infinity*, this set T does not exist, so the paradox is not true. Similarly, the *object* (namely, *all Turing machines*, including those that have not yet been generated) for the Turing Machine Halting Problem is also a *completed all* and an *actual infinity*, which leads to the Turing machine that we supposed to judge whether it is itself halted, resulting in contradictions.

However, **the Judgment Procedure Argument** is not intended to recognize the existence of such a Turing machine to judge *all Turing machines;* instead, it is to deny its existence. We think we can list all the Turing machines one by one (a way of *actual infinity*), but we can always design new Turing machines outside of the existing Turing machines. Therefore, the solution of the Turing Machine Halting Problem, in turn, proved the mistake of the *thought of actual infinity*, but our great Mr. Hilbert and Mr. Turing failed to recognize the nature behind the problem.

The Turing Machine Halting Problem shows that the all-encompassing, omnipotent Turing machine does not exist because it faces an evolving infinite world (that is, a *potential infinity*) rather than a static world (an *actual infinity*). Like Russell's Paradox, the set that ***contains everything that does not belong to itself*** does not exist, because the world of sets is also an evolving infinite world. Therefore, the Turing machine that determines whether all Turing machines are halted does not exist, that is, *undecidable*. The essential idea is that it is impossible to judge and conclude an evolving infinite world. We cannot generate all sets at a point in time, nor can we generate all Turing machines at a point in time. That is, our human understanding of the world is not once and for all, that is, not in *a way of actual infinity*.

5. Doubts on the Proof of the Gödel Incompleteness Theorem

5.1 The Philosophical Significance of Gödel Incompleteness Theorem

We know that the birth of the **Gödel Incompleteness Theorem** has had a great impact on philosophical and scientific circles. The view that is widely accepted by the academic community is that his philosophical contributions mainly focus on the following: We human beings cannot exhaust all the theorems and truths, that is, we cannot generate all the truths and theorems at a certain point in time, and our scientific knowledge of the real world can only be approximate accuracy rather than absolute accuracy. However, this philosophical meaning also hides another expression of philosophical meaning, that is, the idea of *actual infinity* (the thought of exhausting all truths) that humans uphold is wrong; that is, the **Gödel Incompleteness Theorem** proves that the epistemology philosophy of *actual infinity* is not valid. This point is precisely in line with Hegel's thought of *dialectical infinity*, that is, in line with the epistemology of dialectical materialism. Therefore, the greatest discovery of Gödel's life was to deny the idea of *actual infinity* by using the method of *actual infinity* and thus completely deny himself.

5.2 Master of Philosophy, Wittgenstein questioned the proof of the Gödel Incompleteness Theorem

Once the **Gödel Incompleteness Theorem** was put forward, it attracted great philosophical debate, especially from the philosopher Wittgenstein. As Juliet Floyd, a professor of philosophy at Boston University, said, the philosophical collision between Wittgenstein and **Gödel** was the most fascinating and extreme in the twentieth century. In his *Logic Journey*, Wang Hao detailed the debate between the two. He said, "Gödel emphasizes abstract and general things, while Wittgenstein focuses more on concrete and special things. Gödel was very interested in the relationship between philosophy and science. In Wittgenstein's view, *'the difficulty of philosophy is to say only what we know'*. **.......** Although they both believe that philosophy is conceptual research, they have completely different views on concepts. Although both put considerable effort on the philosophy of mathematics, their perspectives and conclusions are often opposite" (Wang, 2009, pp. 224-225). In Gödel's view, Wittgenstein does not understand the incompleteness theorem, and Gödel said, "He interpreted it as a logical paradox that, but in fact, on the contrary, is a mathematical theorem of an undisputed part of mathematics (limited number theory or combinatorial mathematics)." (Wang, 2009, p. 227) We believe that Wittgenstein's challenge of the incompleteness theorem is well founded and that the "*true but unprovable*" proposition is meaningless. Here, we question it from three aspects.

5.3 Prepare Knowledge

To facilitate the discussion, we first explain the relevant formulas as follows:

The predicate $W(x_1, x_2)$ in the formal system *N* holds if and only if x_1 is the term corresponding to the Gödel number of a *well-formed* formula $B(y)$, and x_2 is the term corresponding to the Gödel number of the proof of $B(x_1)$ in *N*. Its corresponding relationship on the model $\mathbf{D_N}$ is W (p, q), which is a **primitive recursive** relationship. **Therefore, formula $B(y)$ is the basis of the predicate $W(x_1, x_2)$, and it is a definite valid formula, not an invalid formula. [This is something we have not fully noticed before, and it is the key to Gödel's proof.**

We know that the relation W (p, q) on the model $\mathbf{D_N}$ is primitive recursive if and only if the relationship is expressible in the formal system *N*, which is the basis for the proof of the **GIT (Gödel's Incompleteness Theorem)**. **Therefore, we can also say that the predicate $W(x_1, x_2)$ is also primitive recursive, which is very important and places effective restrictions on its domain.**

In the proof of the **Gödel Incompleteness Theorem**, the *undecidable formula* $u = A(0^{(p)}) = (\forall x_2)\neg W(0^{(p)}, x_2)$ is called the **Gödel formula**. (A. G. Hamilton, 1989, pp. 192-194), where p is the Gödel number of the formula $A(x_1) = (\forall x_2)\neg W(x_1, x_2)$, expressed with the function g, we have $p = g(A(x_1))$. $0^{(p)}$ represents the term corresponding to the number p.

We use *wf.* to represent '*well-formed formula'*.

Regarding **Russell's Paradox**, its general expression form is $T = \{x \mid x \notin x\}$. The meaning is as follows: T is a set made up of all sets that do not belong to themselves.

Γ is a set of *well-formed* formulas for the first-order language *L*. The model of *Γ* refers to the following: An interpretation of *L* is called the model of *Γ* if in this interpretation every element of *Γ* is true.

The model of a first-order system *S* refers to an interpretation of *S* called the model of *S*, and if in this interpretation, every theorem of *S* is true.

Question-1: The basis of the proof of the GIT is the idea of *actual infinity*. In the proof of the GIT, the relation W (p, q) and predicate $W(x_1, x_2)$ do not agree; the object of the former is a *potential infinity*, while the object of the latter is an *actual infinity*. This clearly contradicts the "*expressible relationship*" between them. Therefore, the Gödel formula does not belong to the effective definition domain of the predicate $W(x_1, x_2)$ and cannot be used to prove the GIT.

We believe that the *undecidable* **formula *u* (i.e., the Gödel formula) is not a *logically valid* formula and should be outside the valid domain of the predicate $W(x_1, x_2)$ of the formal system *N*;** there are always some *well-formed* formulas that are outside the valid domain of the predicate $W(x_1, x_2)$. Such formulas exist in form but not in interpretation, creating a kind of idling. It can also be said that the predicate $W(x_1, x_2)$ that decides all the *well-formed* formulas does not exist (otherwise it must be restricted). This is exactly the same as the T in Russell's Paradox (the so-called *set composed of all sets that do not belong to themselves*). That is, we cannot exhaust all the Turing machines, all the sets, and all the *well-formed* formulas.

The relation W (p, q) holds if and only if p is the **Gödel** number of formula $wf.B(x_1)$, where x_1 appears freely in $wf.B(x_1)$, and q is the Gödel number of the proof of $wf.B(0^{(p)})$ in the formal system *N*. Relation W (p, q) is *primitive recursive* in the model $\mathbf{D_N}$ (such as the set of natural numbers), which is a kind of ***potential infinity***, while in the formal system *N*, "*all the well-formed formulas*" is a kind of ***actual infinity***. The object that the predicate $W(x_1, x_2)$ is going to process is a "*completed all*" and an *actual infinity*. This leads to $W(x_1, x_2)$ determining a certain kind of *well-formed formula* defined by itself, which leads to contradictions, hence leading to the emergence of the undecidable *well-formed* formula $u = A(0^{(p)}) = (\forall x_2)\neg W(0^{(p)}, x_2)$.

Here, there is a situation completely similar to **Russell's Paradox**, and the Gödel formula *u* is similar to $T = \{x \mid x \notin x\}$ in **Russell's Paradox**. The predicate determines a *well-formed* formula composed of itself; the essence lies in that we insist on the idea of *actual infinity*. In the opinion of Wang Hao, a famous mathematical logician, the main reason Wittgenstein opposes **Gödel**'s proof is this *actual infinity*. He pointed out, "The *'true but unprovable'* proposition assumes that positive integers or the infinite set of P (1), P (2), etc., act as a complete whole, not the potential whole" (Fan, 2017, p. 172). Therefore, we believe that the basis of the proof of the **GIT** is a complete idea of *actual infinity*.

Since the *actual infinity* is different from the *potential infinity*, the relation W (p, q) and the predicate $W(x_1, x_2)$ must be equivalent, and this requires that the object that the predicate $W(x_1, x_2)$ processes should not be an *actual infinity*. The former is *primitive recursive*, an impossible self-loop, while the object processed by the latter is an *actual infinity*, which must contain its own judgment of itself, and must appear self-loop, thus destroying its own *primitive recursiveness*. Therefore, the Gödel formula is bound to be excluded from the effective definition domain of the predicate $W(x_1, x_2)$; that is, the Gödel formula does not belong to the effective definition domain of the predicate $W(x_1, x_2)$, and the Gödel formula cannot be used to prove the GIT.

As we know, not all the well-formed formulas of the formal system *N* can be used to represent a **number-theoretic function**, such as formula $x_1 = x_1 \wedge y \neq y$ is impossible to represent a **number-theoretic function**; and the Gödel formula is such a kind of well-formed formula, so it does not belong to the effective definition domain of predicate $W(x_1, x_2)$.

Question-2: The Gödel formula *u* is a circular description, not a definite, valid formula. The basis of the predicate $W(x_1, x_2)$, the formula $B(y)$ corresponding to the variable x_1, must be a definite effective formula. The contradiction of the proof originates from the Gödel formula itself, which cannot be attributed to the incompleteness of the system; thus, the proof is wrong.

The formula described by a cycle is not an effective formula; its Gödel number does not belong to the valid domain of relation W (m, n), and it cannot be used to prove the GIT. We know that the Gödel formula *u* is the premise of the GIT proof, and if it is an invalid formula, then the GIT proof is also invalid.

In formula $A(x_1) = (\forall x_2)\neg W(x_1, x_2)$, x_1 is the term corresponding to the Gödel number of a **definite** formula $B(y)$, and we substitute x_1 into formula $B(y)$ to obtain $B(x_1)$. The significance of the above $wf.A(x_1)$ is that formula $B(x_1)$ cannot be proven in system *N*. Since we do not limit what this formula $B(x_1)$ is, it can be formula $wf.A(x_1)$ itself. In this way, formula $wf.A(x_1)$ buries the seed of the cyclic definition in its definition, that is, formula $A(x_1)$ can be interpreted as follows: **formula $\underline{A(x_1)}$ cannot be proved in system *N***, thus becoming an unknown cycle object. **We define arbitrary formulas by using this "*unlimited*" predicate $W(x_1, x_2)$, which is exactly like the definition of set T in Russell's Paradox.** This shows that the predicate $W(x_1, x_2)$ crosses a certain boundary and goes to its opposite.

Indeed, formally, formula $wf.A(x_1)$ or formula *u* is understandable, and they are all *well-formed* formulas. Similar to the definition of T in Russell's Paradox, there is no problem in form, but the explanation is ineffective. **The Gödel formula *u* is *undecidable* in system *N*, which shows only that formal system *N*, such as the concept of "*set*", also needs to be limited.**

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For formula $A(x_1) = (\forall x_2)\neg W(x_1, x_2)$, we replace x_1 at the right end of the above formula with **0^(p)**, and we obtain $A(x_1)=A(0^{(p)}) = (\forall x_2)\neg W(0^{(p)}, x_2)$. Then, we put $p=g(A(x_1))$ into the above formula, and we obtain $A(x_1)=A(0^{(p)}) = (\forall x_2)\neg W(0^{(g(A(x1)))}, x_2)$, that is, $A(x_1) = (\forall x_2)\neg W(0^{(g(A(x1)))}, x_2)$.

Obviously, formula $A(x_1)$ is a circular definition, forming an infinite backward chain, which is its own basis, although we disguise it by the Gödel number (or Gödel mapping). No one knows exactly what formula $A(x_1)$ truly is. This expression is similar to the infinite cycle of '*a set belongs to oneself'* in Russell's Paradox. **This obviously violates the basic requirement of W (p, q) or predicate $W(x_1, x_2)$, namely, that the *well-formed* formula $B(y)$ corresponding to x_1 must be a definite valid formula rather than a circular expression.**

Under the circular definition, the validity of the Gödel formula ***u*** does not hold; that is, ***u*** is not a valid formula. A logical invalid formula has no right or wrong points, and it becomes meaningless to take it as an "*undecidable formula*". Thus, the emergence of ***u*** is not due to the incompleteness of the formal system ***N*** but due to the definition of ***u*** itself. **This is the fundamental reason why the great Wittgenstein considered Gödel's proof as a logical paradox.**

Since ***u*** is an invalid formula, we believe that any *language L* of the formal system *N* should not include this formula; that is, there is no interpretation of *u* in the model $\mathbf{D_N}$, so there is no so-called '***true but unprovable***' proposition. Otherwise, no suitable model can exist for any language containing *u*, where we must exclude *u* from the normal language or some restriction on our formal language.

As Wittgenstein said, *tautology* and *contradictory* logical propositions are meaningless, and they have no rich connotations, which means that nothing is said about the world. Therefore, as Juliet Floyd points out, "Wittgenstein's fundamental view on Gödel's proof is that he showed the impossibility of some kind of construction—just as it is impossible with a ruler and a compass to divide an angle in three parts. Wittgenstein warns people against trying to find the derivability of these sentences. Indeed, we will insist that the derivability of such a sentence is not a derivability in the relevant sense. (Floyd, 1995)

Question-3: The logical approach of the proof of the GIT is completely different from the proof of the Turing Machine Halting Problem and the solution of Russell's Paradox.

The scientific responses to **Russell's Paradox** and **Turing Machine Halting Problem** are that humans deny the existence of this contradiction rather than affirming its existence. In contrast, the proof of the GIT recognizes the rationality of this contradiction, namely, the rationality of the **Gödel** formula, rather than denies it.

As above, we know that the existence of the *undecidable formula* ***u*** in the proof of the GIT is no different from the existence of T in Russell's Paradox. This is also completely similar to the Turing Machine Halting Problem, but we have come to the opposite conclusion. From the solution of the

Turing Machine Halting Problem, we conclude that there is no Turing machine to judge *all Turing machines*. From the GIT, we conclude that the form system *N* is incomplete, rather than questioning the rationality of the *domain* of the predicate $W(x_1, x_2)$ that decides *all the well-formed formulas*, that is, questioning the rationality of the Gödel formula itself. **This unrestricted predicate $W(x_1, x_2)$ is similar to the *relationship of belonging* in Russell's Paradox.**

We argue that the domain of the predicate $W(x_1, x_2)$ is limited, just as is our solution to **Russell's Paradox**: The definition of a set must be limited. That is, **p** does not belong to a valid domain of the relation W (m, n). This formula ***u*** is the same as the definition of set T in Russell's Paradox, which is a circular definition. The essence of the circular definition is that the idea of *actual infinity* works; that is, **we judge the *"things"* that have not been generated as *"things that already exist"*, thus leading to contradictions.**

Below, we compare the formula ***u*** to the set T in Russell's Paradox.

We set *M* as the set of all theorems of the formal system ***N***. In the proof of the GIT.

If $\mathbf{u} \in M$, the relationship W (p, q) holds. We derive that both $W(0^{(p)}, 0^{(q)})$ and $\neg W(0^{(p)}, 0^{(q)})$ are theorems for formal systems ***N***, which contradicts the consistency of ***N***, and thus $\mathbf{u} \notin M$.

However, if $\mathbf{u} \notin M$, then the relationship W (p, q) does not hold; based on the ω - consistency, we conclude that $\neg \mathbf{u} \notin M$. In this case, although for each *q*, there is $\neg W(0^{(p)}, 0^{(q)})$ holds, but we cannot confirm whether $\mathbf{u} \in M$ is true, that is, we cannot confirm whether $(\forall x_2)\neg W(0^{(p)}, x_2) \in M$ is true; it may or may not be true. Under the different models, we will have the different results. Under the Standard Model D_N (namely, the Normal Model), only the interpretations of 0, $0^{(1)}$, $0^{(2)}$, $0^{(3)}$, $0^{(4)}$, … are included, and there is no interpretation for the other elements. In this case, for each *q*, there is $\neg W(0^{(p)}, 0^{(q)})$ holds, then we have $(\forall x_2)\neg W(0^{(p)}, x_2) \in M$, that is, $\mathbf{u} \in M$ is true.

Therefore, under the Standard Model D_N, we can derive $\mathbf{u} \notin M$ from $\mathbf{u} \in M$; from $\mathbf{u} \notin M$ we can also derive $\mathbf{u} \in M$.

While in Russell's Paradox, $T = \{x \mid x \notin x\}$, we have：

If $T \in T$, according to the definition rules, T is its own element, and we obtain $T \notin T$;

If $T \notin T$, then T meets the definition of the element of T and thus has $T \in T$. This leads to a conflict.

A comparative analysis of these two proofs obviously shows that they are very similar, and the former is not completely equivalent to the latter, but the former contains the possibility of the latter. **That is, under the Standard Model D_N, the Gödel formula is a typical Russell's Paradox,a self-contradiction.**

We can clearly analyze this cycle. In **Russell's Paradox**, the treatment is that, as a set, such a T does not exist. In the proof of the **Turing Machine Halting Problem**, people also did not recognize the existence of the Turing machine H that judges *"all Turing machines"*. We therefore hold that this *undecidable formula* ***u*** constructed in the proof of GIT, as a contradiction, should not exist and should not be admitted. We should reflect on the other possibilities of the problem as we deal with Russell's Paradox, rather than considering the existence of ***u*** as reasonable and thus concluding the formal system *N* as incomplete. In Gödel's view, the rationality of the existence of ***u*** is based on the rationality of its construction; however, similarly, the construction of T in Russell's Paradox is also reasonable, but we do not admit the rationality of the existence of **T;** instead, we reflect on whether the definition rule of the set itself is reasonable.

Therefore, the GIT proof shows that its logical approach is obviously different from the former two. **We believe that *u* is a formula with invalid semantics and invalid logic, with its correct form and invalid content, similar to T in Russell's Paradox, which cannot be used to prove the GIT.** As Wittgenstein questions the meaning of ***u***; he claims that it is not a meaningful mathematical proposition, that ***u*** is not available in mathematical proofs and calculations and that we should abandon the natural language interpretation of ***u***. Wittgenstein points out, "Mathematical propositions, like other propositions, do need to explain their grammar" (Wittgenstein, 2003, p. 291). That is, we must regulate and restrict mathematical propositions such as Russell's Paradox and the Gödel formula.

6. Summary

Above, we have questioned the reasonableness of the proof of the **GIT** from three aspects, and we believe that the definition and use of the Gödel formula should be abandoned. Not as Gödel thinks—the system is incomplete, the so-called *'true but unprovable'* proposition does not exist. Just as in Russell's Paradox, what we want to limit is the definition of sets, so the set ***that contains all sets that do not belong to themselves*** does not exist. Gödel's proof was based on an invalid formula and was therefore erroneous.

The world is constantly developing and changing, as Engels said, "Nature does not exist, but generated and disappear." Russell's Paradox, the Turing Machine Halting Problem, and GIT all illustrate the truth that our world moves forward in a way of *potential infinity*. The world is ultimately Aristotelian, not Platonic. Let us return to the development of *dialectical infinity*.

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