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Original Paper

Who is lying? Wittgenstein or Gödel?

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Abstract

Gödel's Incompleteness Theorem *was a major discovery in modern logic that has consistently attracted the attention of scientific and philosophical circles. However, since* ***Gödel's Incompleteness Theorem*** *was put forward, the scientific and philosophical significance of its proof has been questioned, especially Wittgenstein, who strongly criticized it. Wittgenstein's main argument against this theorem was that he believed* ***Gödel sentence*** *was a logical contradiction, that there was no such thing as a* ***true and unprovable*** *proposition at all, that it was a self-referential and self-contradictory concept, and that we should abandon the natural language explanation of this proposition. The author points out that* ***Gödel sentence*** *is a logically invalid formula constructed outside the domain of* ***the principle of representability*** *and lacks recursion; this contradiction stems from the abuse of formalization in language. At the same time, the author discovered a paradox in the proof of* ***Gödel's Incompleteness Theorem****, that is, Gödel sentence itself is a kind of Russell's Paradox. The author believes that the philosophical root of Russell's Paradox and* ***Gödel sentence*** *lies in adhering to the idea of actual infinity, while the* ***Infinite Exchange Paradox*** *overturns the* ***Cantor's diagonal argument****, which shakes the foundation of the idea of actual infinity. The above findings strongly support Wittgenstein's view. Therefore, the conclusion of this paper is that Wittgenstein's questioning of Gödel sentence is completely correct, the proof of* ***Gödel's Incompleteness Theorem*** *is based on a logically invalid formula, and the contradiction of the proof originates from* ***Gödel sentence*** *itself, thus it cannot be attributed to the incompleteness of the system, so the proof is wrong.*

Keywords

Russell's Paradox, Gödel's Incompleteness Theorem, Gödel sentence (Gödel's proposition), Infinite Exchange Paradox, Cantor's diagonal argument, the principle of representability.

It is well known that in 1901, Russell discovered **Russell's Paradox**, which brought the third crisis to the development of mathematics. The solution to this paradox remains undetermined to this day. Soon after, in 1931, the logician Gödel proved the famous incompleteness theorem, which induced an epoch-making change in the study of the basis of mathematics. However, this theory has been strongly criticized by the expert philosopher Wittgenstein. Although the circles of mathematics and logic tend to fully accept this theory, the debate between the two ideas has not been resolved. However, Gödel based his proof on an invalid formula, which makes Gödel's proof meaningless. Let us analyze it in detail below.

1. Master of Philosophy, Wittgenstein Questioned the Proof of Gödel's Incompleteness Theorem

Once **Gödel's Incompleteness Theorem** was put forward, it attracted great philosophical debate, especially from the philosopher Wittgenstein. As Juliet Floyd, a professor of philosophy at Boston University, said, the philosophical collision between **Wittgenstein** and **Gödel** was the most fascinating and extreme in the twentieth century. In his *A Logic Journey*, **Wang Hao** detailed the debate between the two. He said, "Gödel emphasizes abstract and general things, while Wittgenstein focuses more on concrete and special things. Gödel was very interested in the relationship between philosophy and science. In Wittgenstein's view, *'the difficulty of philosophy is to say only what we know'*. **…….** Although they both believe that philosophy is conceptual research, they have completely different views on concepts. Although both put considerable effort on the philosophy of mathematics, their perspectives and conclusions are often opposite" (Wang Hao, 2009, pp. 224-225). In Gödel's view, Wittgenstein does not understand the incompleteness theorem, and Gödel said, "He interprets it as a kind of logical paradox, while in fact it is just the opposite, namely a mathematical theorem within an absolutely uncontroversial part of mathematics (finitary number theory or combinatorics)." (Wang Hao, 2009, p. 227) We believe that Wittgenstein's questioning of the incompleteness theorem is well-grounded, and the proposition of "*true but unprovable*" is meaningless. Below, we raise our doubts from two aspects.

2. Prepare Knowledge

To facilitate the discussion, we first explain the relevant formulas as follows:

The predicate $W(x_1, x_2)$ in the formal system ***N*** (such as Russell's PM system) holds if and only if x_1 is the term corresponding to the Gödel number of a *well-formed* formula $B(y)$, and x_2 is the term corresponding to the Gödel number of the proof of $B(x_1)$ in *N*. Its corresponding relationship on the model D_N is W(*p, q*), which is a ***recursive*** relationship. **Therefore, formula $B(y)$ is the basis of the predicate $W(x_1, x_2)$, and it is a definite valid formula, not an invalid formula. [*This is something we have not fully noticed before, and it is the key to Gödel's proof.*]**

We know that the relation W(*p, q*) on the model D_N is ***recursive*** if and only if the relationship is ***representable*** (or ***expressible***) in the formal system *N*, which is the basis for the proof of **Gödel's Incompleteness Theorem (for short, GIT)**. **Therefore, we can also say that the predicate**

$W(x_1, x_2)$ **is also *recursive*, which is very important and places effective restrictions on its domain. Neither *representability* nor *Gödel mapping* (or Gödel numbering) changes the *recursiveness* of a primordial relation, function or proposition.**

This *recursiveness* or *the principle of representability* requires that when we discuss, analyze, and reason about all the *well-formed* formulas in *N* (including logical reasoning), we must limit all our "*logical reasoning behavior*" to this valid range, that is, the *well-formed* formula discussed is limited to the range required by this *recursiveness*. In other words, any *well-formed* formula formed by the predicate $W(x_1, x_2)$ must be within the range required by this *recursiveness*, and any logical reasoning and logical calculation that violates the requirement of *recursiveness* must violate *the principle of representability*.

This requirement of *recursiveness* defines a *space* for our logical reasoning activities, and this *space* is the *effective domain* of our logical activities, that is, the *boundary* of our activities. The essence of *recursiveness* is that any calculation (including logical reasoning, logical calculation) must be completed in a finite number of steps, and any kind of circular calculation (including logical calculation) does not meet the requirements of *recursiveness*.

In the proof of **Gödel's Incompleteness Theorem**, the *undecidable formula* $u = A(0^{(p)}) = (\forall x_2)\neg W(0^{(p)}, x_2)$ is called **Gödel sentence** (A. G. Hamilton,1989, pp192-194), where *p* is the Gödel number of the formula $A(x_1) = (\forall x_2)\neg W(x_1, x_2)$, expressed with the function g, we have $p = g(A(x_1))$. $0^{(p)}$ represents the term corresponding to the number *p*. The meaning of **Gödel sentence** is: "*Formula* $A(0^{(p)})$ *is unprovable*", and $A(0^{(p)})$ is the **Gödel sentence *u*** itself. Therefore, the meaning of **Gödel sentence** is also interpreted as: "*I am unprovable*" or "*This formula is unprovable*", thus forming a semantic paradox. However, this semantic paradox is essentially a logical paradox, which will be explained later (Please refer to Chapters 5 and 6).

We use *wf.* to represent '*well-formed formula'*.

Regarding **Russell's Paradox**, its general expression form is $T = \{x \mid x \notin x\}$. The meaning is as follows: ***T*** is a set made up of all sets that do not belong to themselves.

3. Actual Infinity, Dialectical Infinity, Infinite Exchange Paradox and Cantor's Diagonal Argument

According to the understanding and usage habits of mathematicians, the concepts of *actual infinity* and *potential infinity* can be defined as follows:

Actual infinity: It is a completed state (gone). For *an infinite process*, it definitely reaches the ultimate point of the process, and the enumeration procedure for *an infinite process* can definitely be completed; acknowledging the existence of "*infinity*" as a whole, such as the existence of infinite sets. Obviously, Cantor's set theory and Cantor's diagonal argument belong to the realm of *actual infinity*. Any judgment made regarding the "entire members" of *infinity* belongs to *actual infinity*. For instance, the predicate *W* used in the proof of Gödel's incompleteness theorem involves a judgment about "*all well-formed formulas*" (an infinite set).

Potential infinity: It is an ongoing state (going). It denies the attainment of the ultimate point of *an infinite process*, and thus the enumeration procedure for *an infinite process* cannot be completed; it denies the existence of *infinity* as a whole, and therefore does not recognize the existence of infinite sets. Obviously, *recursiveness* is a kind of *potential infinity*.

In short, *actual infinity* and *potential infinity* have taken completely opposite positions on these two issues ------ "*Can the infinite process come to an end (completeness)?*", "*Does the totality of infinity exist (existence)?*". Generally speaking, formalists and logicalists adhere to the concept of *actual infinity*, while intuitionists uphold the concept of *potential infinity*. As for mathematicians and philosophers, Plato, Russell, Hilbert, Cantor, Gödel, etc. adhered to the concept of *actual infinity*, while Aristotle, Poincaré, Kronecker, Brouwer, Wittgenstein, etc. upheld the concept of *potential infinity*.

In the article *Philosophical Infinity and Mathematical Infinity* (Zhang Hong, Zhuang Yan, 2019), the author raised the *Infinite Exchange Paradox*, rejected *Cantor's Diagonal Argument*, and comprehensively introduced Hegel's concept of *dialectical infinity*. In view of this, the author proposed *the view of dialectical infinity*, which means: acknowledging the existence of *infinity* as a whole (or a subject), but also acknowledging that *the infinite process* is impossible to complete, that is—*infinity* exists but cannot be completed. The author believes that "***the totality of infinity***" and "***the infinite process***" belong to different categories, just as "a straight line (or a line segment)" and "a point" belong to different categories; therefore, they cannot be substituted for each other. *Infinity* exists but cannot be completed. The existence of *infinity* does not equate to its completeness. ***The infinite process*** is essentially a human subjective behavior and thus is inherently unachievable. The so-called "*infinite process can be completed*", but that is merely a *completion* of our human subjective imagination. Infinity as a process is an eternal movement and is always in a state of motion. Human cognition (whether through mathematics or physics) can only be in the process of *approaching infinity*, and can never *terminate* this process. The objective existence of *infinity* is precisely manifested in the never-ending nature of *the infinite process*; if *the infinite process* is completed, then it becomes a finite one. This is because *the infinite process* can never be completed, and it is precisely for this reason that it is considered *infinity*. *The view of dialectical infinity* not only acknowledges the objective existence and knowability of *infinity*, but also recognizes the incompleteness of *the infinite process*. That is to say, human's understanding of infinity is a dialectically developing process.

On the completion of *an infinite process*, Engels denied directly in *Anti-Dühring*. He said, "It is just the same with eternal truths. If mankind ever reached the stage at which it should work only with eternal truths, with results of thought which possess sovereign validity and an unconditional claim to truth, it would then have reached the point where the infinity of the intellectual world both in its actuality and in its potentiality had been exhausted, and thus the famous miracle of the counted uncountable would have been performed." (Engels, 1947, p. 129)

The modern idea of actual infinity* not only acknowledges the existence of *infinity*, but also believes that *the infinite process* can be completed.** In fact, it confuses "existence (as a whole)" with "process". *Infinity* as a whole, as an existence, is not the outcome of a process, but rather a pre-existing (or agreed-upon) structure. For instance, in set theory, the set of natural numbers N is regarded as a complete entity. This is an ***agreement regarding the *infinite entity* of natural numbers, rather than the result obtained by counting. The consequence of this confusion is the *Infinite Exchange Paradox* and other contradictory problems such as "*the tortoise and the hare race*" and "*the stationary arrow*". This idea conflates "*existence of infinity*" (objective) with "*completion of the infinite process*" (subjective), resulting in a logical "*spooky action at a distance*"—that is, thinking can instantaneously grasp *an infinite whole*, thereby ignoring *the finite-infinite contradiction* contained within the infinite process itself. Therefore, the contradictions and chaos brought about by *the idea of actual infinity* are essentially a conflict between metaphysics (which regards "infinity" as a static completed entity) and dialectics (which regards "infinity" as an eternal process of movement).

As we know, Mr. Hegel's greatest contribution is to propose the idea of dialectic, especially the idea of *dialectical infinity* (*real infinity* and *bad infinity*) in mathematics. Hegel believes that the infinite thing has a double meaning, infinite thing is the unity of *bad infinity* and *real infinity*; not only confirms that *bad infinity* is a basic form of *infinity*, but also criticizes the one sidedness, which only talks about *bad infinity* and does not pay attention to *real infinity*. So *infinity* is the unity of *bad infinity* and *real infinity*; *Real infinity* cannot be separated from *bad infinity*, *Being-for-self* is inseparable from *Being-in-itself*. Human's understanding of *infinity*, from possible to reality, from the abstract to the concrete, has completed the transformation from *bad infinity* to *real infinity*. From the perspective of the cognitive process, it can be said that *bad infinity* is the result of people's understanding of *infinity* from the outside of things and from phenomena; *Real infinity*, on the other hand, is the product of delving into the essence of things, the universal connections within them, and essentially understanding *infinity*. The transition from *bad infinity* to *real infinity* marks the deepening of human's understanding, reflecting the dialectical development process of human's understanding of *infinity* from possibility to reality, from abstraction to concreteness, and also demonstrating the subjective initiative of human beings in the process of understanding the issue of the infinite.

From *bad infinity* to *real infinity* is an internal transcendence, a dialectical process. *Real infinity* and *bad infinity* are the basic forms of *infinity*. Hegel proposed the profound and dialectical conclusion that *real infinity* encompasses and sublates *bad infinity*. He aimed to specifically and realistically grasp the nature of *infinity* and opposed making abstract inferences about it. The transformation from *bad infinity* to *real infinity* is the transformation from understanding to reason, is a great leap in human's understanding of *infinity*, and is **the highest task of Hegel's philosophy**. Based on this idea, Hegel provided a correct philosophical exposition of the concept of *limit*, revealing the idea of *bad infinity* and *real infinity* in mathematics from *the law of mutual transformation between quality and quantity*, and powerfully criticized the metaphysical trend of thought in mathematics, enabling us to thoroughly understand the essence of *infinity* and providing a reliable philosophical basis for the complete resolution of the Second Mathematical Crisis.

***The View of Dialectical Infinity* holds that *infinity* is the unity of quality and quantity;** *Bad infinity* represents quantity of *infinity* (motion), while *real infinity* represents quality (law, commonness or connection) of *infinity*. *The View of Dialectical Infinity* holds that the *actual infinity* separates the finite from the infinite, and looks at things from the point of view of stillness rather than movement, which has an internal irreconcilable contradiction, and is a metaphysical idealistic view of infinity. The concept of *actual infinity* is exactly the same thing as Transcendentalism, which in essence believes that the development of the world, the movement will have an end. However, Hegel's view of *Infinity* is not only to see *the universal connectivity of the objective world (real infinity)*, but also the objective reality of "*the infinite process cannot be completed (bad infinity)*", so it is a scientific view of infinity, but also a view of *dialectical infinity*.

In *the view of dialectical infinity*, *infinity* is an objective reality. For instance, time and space are naturally objective infinities, and the immortality of matter is also an objective infinity. This confirms the rationality of the existence of an infinite set. At this point, *the view of dialectical infinity* is consistent with the view of *actual infinity*, both confirming the rationality of the existence of *infinity*. However, the main difference between their ideas is on the view of "*whether the infinite process can be completed*", *the view of dialectical infinity* believes that the *infinite process* as the main body understanding object process, as the contradiction between *finity and infinity*, is impossible to complete, impossible to end, but *the view of actual infinity* thinks that *infinite process* can be completed, end, abandoned the contradiction between *finity and infinity*, with subjective instead of objective, imposed the product of thinking on the objective material world, thus is a kind of idealistic epistemology. ***The existence of infinity* and *the impossibility of infinite process* are two completely different concepts, belonging to different categories and cannot be substituted for each other.** *Dialectical infinity* is not simply *counting up*, but rather grasping its inherent inevitability and connections within the process of infinite development. Therefore, the view of *dialectical infinity* does not oppose Cantor's theory of infinite sets, but rather negates his diagonal argument.

The thought of *actual infinity* in modern mathematics (such as Cantor's theory of transfinite numbers), fundamentally lies in substituting the *impossibility of an infinite process* with the *completeness (existence) of the whole*, thereby confusing "the process" with "the entity". Objectively, the natural numbers as a sequence are always being generated (1, 2, 3, ...), which is a process of *potential infinity* and will never come to an end. Subjectively, for convenience, mathematicians use the symbol ***N*** to represent this unfinished whole and view it as an actual "*infinity*", which is actually a kind of "*agreement*". The consequence of this is that this operation of forcibly treating "*unfinished processes*" as "*completed entities*" is precisely the breeding ground for Russell's Paradox, Cantor's diagonal argument paradox (i.e., the *Infinite Exchange Paradox*) and Gödel's self-referential sentence. **Because when you treat something that is fundamentally a "process" as**

an "object", you introduce logical type confusion. For instance, when Gödel constructed the formula *u*, he actually presupposed an "*actual infinitude*" traversal for all possible proofs, and this implicitly contained a "*vicious circle*". A detailed explanation will be provided later.

The *Infinite Exchange Paradox* proposed by the author fundamentally challenges the legitimacy of the concept of *actual infinity*, providing a completely new critical perspective for resolving *the issues of infinity* in mathematics and philosophy, and strongly supporting Wittgenstein's viewpoint. The core logic of the *Infinite Exchange Paradox* lies in revealing the fundamental contradiction of the view of *actual infinity* in the assumption of "*completing the infinite process*". The paradox demonstrates through dynamic replacement of elements in an infinite set that any operation attempting to exhaust the infinite set is impossible. This paradox not only refutes ***Cantor's diagonal argument***, but also poses a challenge to a series of mathematical proofs based on the concept of *actual infinity*, including the proof basis of Gödel's Incompleteness Theorem.

We know that Wittgenstein strongly opposed *Cantor's diagonal argument* and conducted a profound analysis and criticism of this erroneous idea. He denied the reality of infinity. He believed that the reality of infinity could not be proved, and that **symbols could not express the reality of infinity.** As he said in *Research in Basic Mathematics*, "it says that *actual infinity* cannot be grasped by mathematical symbolic systems at all, so it can only be described and not shown. This description may have been grasped in a way similar to the following: a large number of things that cannot be held in hand by people are lifted by packing them in boxes." (Wittgenstein, 2013a, p. 210) He opposes the use of *Cantor's Diagonal Argument*, arguing that *infinity* cannot be exhausted by *finity*. He said: "Because we have the following legitimate feeling: where the last thing can be talked about, there cannot be '*there is no last thing at all*'." (Wittgenstein, 2013a, p. 207). **He holds that infinity and finity are completely different categories, and infinity is an inherent stipulation.** He believes that infinity is not a number, and infinity is not a magnitude that competes with finiteness; rather, it is an inherent characteristic. He pointed out that, "*Infinite sets* and *Finite sets* are two different logical categories. Something that can be meaningfully expressed to one category cannot be meaningfully expressed to another category." (Wittgenstein, 2013a, p. 206) Taking the number π as an example, the number π expresses an infinite law which is accompanied by the actual observation; that is, the number π is a rule, which essentially embodies Hegel's concept of *real infinity*.

4. The Principle of *Representability*

We know that the proof of **GIT** is premised on *the principle of representability*. A relation (or function) that is *representable* in a formal system *N* is equivalent to being *recursive*. The *representability* of a relation is defined as follows:

The *k*-element relation R on the set of natural numbers D_N is *representable* (or *expressible*) in the formal system *N*, which means: If in *N* there exists a well-formed formula $A(x_1, x_2, \ldots, x_k)$ with k free arguments, such that for any $n_1, n_2, \ldots, n_k \in D_N$, there is

(1) If the relation $R(n_1, n_2, \ldots, n_k)$ holds on D_N, then there is $N \mapsto A(\mathrm{O}^{(n_1)}, \mathrm{O}^{(n_2)}, \ldots, \mathrm{O}^{(n_k)})$; That is, the formula $A(\mathrm{O}^{(n_1)}, \mathrm{O}^{(n_2)}, \ldots, \mathrm{O}^{(n_k)})$ is a theorem for *N*.

In addition,

(2) If the relation $R(n_1, n_2, \ldots, n_k)$ does not hold on D_N, then there is $N \mapsto \neg A(\mathrm{O}^{(n_1)}, \mathrm{O}^{(n_2)}, \ldots, \mathrm{O}^{(n_k)})$; That is, the formula $\neg A(\mathrm{O}^{(n_1)}, \mathrm{O}^{(n_2)}, \ldots, \mathrm{O}^{(n_k)})$ is a theorem for *N*.

In this case, we say that the relation $R(n_1, n_2, \ldots, n_k)$ is *representable* in *N* by the formula $A(x_1, x_2, \ldots, x_k)$.

From the above definition, we can clearly know the following facts: a relation is *representable* if and only if its corresponding well-formed formula *A* is either a theorem, or the negative form of that formula is a theorem, *it has to be one or the other*. Under *the principle of representability*, it is impossible for a formula to be both *provable and unprovable*. The *principle of representability* also limits the domain of this formula.

5. Review and Analysis of Wittgenstein's Comments on Russell's Paradox and Gödel Sentence

5.1 Regarding Russell's Paradox

Russell's Paradox had a profound influence on Wittgenstein, and the core of *Tractatus Logico-Philosophicus* (*TLP*, for short) lies in solving the problem of Russell's Paradox. He conducted a comprehensive philosophical reflection on Russell's paradox and believed that he had resolved this contradiction (that is, a method of "*dissolution*"). In my opinion, the fundamental reason why he was so dissatisfied with Russell's "*Introduction*" (*TLP*) lies in the fact that he was a potential infinite while Russell was an actual infinite.

Wittgenstein did not think that Russell's Paradox needed to be solved, but rather should be *dissolved* [see "Herewith Russell's Paradox vanishes." (Wittgenstein, 2022, p. 57)]. That is, through correct language analysis, its meaninglessness is revealed, thereby avoiding philosophical confusion. He believes that the true philosophical task is not to construct complex theories, but to clarify the boundaries of language and avoid meaningless problems arising from the misuse of language. That is to say, in his view, Russell's Paradox is a meaningless problem caused by our human misuse of language.

He later evaluated Russell's Paradox as "*a cancerous growth*". He said, "The Russellian contradiction is disquieting, not because it is a contradiction, but because the whole growth culminating in it is a cancerous growth, seeming to have grown out of the normal body aimlessly and senselessly." (Wittgenstein, 2014, p. 166e)

He opposes a proposition that states to itself. He said, "No proposition can say anything about itself, because the propositional sign cannot be contained in itself (this is the 'whole theory of types')." (Wittgenstein, 2022, p. 57), "A function cannot be its own argument, because the function sign already contains the prototype of its own argument and it cannot contain itself." (Wittgenstein, 2022, p. 57)

He believes that a proposition must be a description of a "*reality*" and opposes circular descriptions, because a self-referential description of a cycle cannot be a "*reality*". He pointed out, "The proposition is a picture of reality, for I know the state of affairs presented by it, if I understand the proposition. And I understand the proposition, without its sense having been explained to me.", "The proposition *shows* its sense.", "The proposition *shows* how things stand, *if* it is true. And it *says*, that they do so stand.", "The proposition determines reality to this extent, that one only needs to say 'Yes' or 'No' to it to make it agree with reality." (Wittgenstein, 2022, p. 67); "Reality must therefore be completely described by the proposition.", "A proposition is the description of a fact.", "The proposition constructs a world with the help of a logical scaffolding, and therefore one can actually see in the proposition all the logical features possessed by reality *if* it is true.", "To understand a proposition means to know what is the case, if it is true." (Wittgenstein, 2022, p. 67), "The proposition communicates to us a state of affairs, therefore it must be *essentially* connected with the state of affairs. And the connexion is, in fact, that it is its logical picture. The proposition only asserts something, in so far as it is a picture." (Wittgenstein, 2022, p. 69); "Propositions can be true or false only by being pictures of the reality." (Wittgenstein, 2022, p. 71)

For a proposition to fully describe a *reality*, it is necessary to impose restrictions on the *referential* of the notation. He pointed out, "The meanings of the simple signs (the words) must be explained to us, if we are to understand them." (Wittgenstein, 2022, p. 69) This kind of restriction is actually to prevent repetitive descriptions. He emphasized that, "The possibility of propositions is based upon the principle of the representation of objects by signs.", "The proposition is a picture of its state of affairs, only in so far as it is logically articulated." (Wittgenstein, 2022, p. 69); "Every proposition must *already* have a sense.", "A proposition presents the existence and non-existence of atomic facts." (Wittgenstein, 2022, p. 75)

To prevent loops, he believed that propositions could only express *reality* but not logical forms. He pointed out, "Propositions cannot represent the logical form: this mirrors itself in the propositions. ...... The Propositions *show* the logical form of reality. They exhibit it." (Wittgenstein, 2022, p. 79)

Therefore, Wittgenstein's solution to Russell's Paradox is essentially based on his idea of *potential infinity*, because the world of *potential infinity* is a constantly developing one, and the idea of *potential infinity* does not allow any self-reference or any cycle. So, he believes that, "The sense of the world must lie outside the world." (Wittgenstein, 2022, p. 183), "He must surmount these propositions; then he sees the world rightly." (Wittgenstein, 2022, p. 189)

5.2 Regarding Gödel Sentence

Although Wittgenstein did not directly refer to **Gödel sentence** as a paradox, nor did he seek specific paradoxical forms existing in the proof of Gödel's theorem (The author will point out in detail later that this paradox does exist), he believed that its self-referential structure would lead to contradictions and that such contradictions stemmed from the formal abuse of language. Thus, it is believed that we should abandon the natural language interpretation of **Gödel sentence**. **Wittgenstein's criticism of Gödel sentence and his criticism of Russell's Paradox are consistent in thought.** Although he did not directly claim that **Gödel sentence** was a paradox, judging from his attitude towards it, he actually regarded it as a paradox.

Wittgenstein did not recognize **Gödel sentence *u*** as a legitimate mathematical proposition, but rather as a pseudo-proposition arising from self-reference or a contradiction in a language game; He believes that a meaningful proposition must point to reality and talk about reality. Wittgenstein pointed out that the construction of *u* relies on natural language explanations (such as "*u cannot be proved*"), which blurs the boundary between formal systems and everyday language and is a "*misuse of language*", thus leading to logical contradictions. In Wittgenstein's view, this self-referent structure reveals the internal contradiction of formal language, but he did not directly characterize this contradiction as a logical paradox, although Gödel himself criticized him for interpreting this formula as a logical paradox. (Wang Hao, 2009, p. 227) He believes that propositions must reflect reality; otherwise, our language will lose its connection with our world. He pointed out, "It is true that one proposition can lead to another, and that one to another, and so on; but in the end we must reach the propositions which do not point to other propositions, but to reality. Or, to put it more precisely: a meaningful proposition refers to reality via the entire chain of definitions". (Wittgenstein, 2015, pp. 271-272); "If that were not the case, then no proposition could be verified. Consequently, there would be no connection between language and reality. "(Wittgenstein, 2015, p. 272)

Wittgenstein believed that *u* does not involve itself, that is, a proposition cannot judge itself. Based on his criticism of Russell's Paradox, he believed that, "**Gödel's proposition**, which asserts something about itself, does not *mention* itself." (Wittgenstein, 2014, p. 176e) (Here, ***Gödel's proposition*** refers to ***Gödel sentence***.) Wittgenstein conducted a detailed analysis of the proof approach of **Gödel sentence**, discussed the "*translation rules*", and analyzed this contradiction. He pointed out, "And let us assume that a rule of translation can be given according to which this arithmetical proposition is translatable into the figures of the first number—the axioms from which we are trying to prove it, into the figures of the other numbers—and our rules of inference into the operations mentioned in the proposition. ----If we had then derived *the arithmetical proposition* from the axioms according to our rules of inference, then *by this means* we should have demonstrated its derivability, but we should also have proved a proposition which, by that translation rule, can be expressed: this arithmetical proposition (namely ours) is not derivable." (Wittgenstein, 2014, pp. 176e-177e)

Wittgenstein believed that *u* is a meaningless symbol, although it has the form of a proposition. He firmly denied the rationality of this **Gödel sentence**, arguing that this logical contradiction should be used with caution and that it should not be applied. He pointed out: "So far the proposition and its proof have nothing to do with any special *logic*. Here, the constructed proposition was simply another way of writing the constructed number; **it had the *form* of a proposition but we don't compare it with other propositions as a sign *saying* this or that, making *sense*.**", "But it must of course be said that that sign need not be regarded either as a propositional sign or as a number sign." (Wittgenstein, 2014, p. 177e); "The various half joking guises of logical paradox are only of interest in so far as they remind anyone of the fact that a serious form of the paradox is indispensable if we are to understand its function properly. The question arises: what part can such a logical mistake play in a language-game?" (Wittgenstein, 2014, p. 179e); "And how could you make the truth of the assertion plausible to me, since you can make no use of it except to do these bits of legerdemain?" (Wittgenstein, 2014, p. 53e); "A contradiction is unusable as such a prediction." (Wittgenstein, 2014, p. 52e)

Wittgenstein believed that meaningful propositions must be verifiable. In his view, if a proposition is meaningful, it must be verifiable. Only when a proposition is verified does it have meaning. He pointed out: "A proposition which cannot be verified in any way is meaningless" (Wittgenstein, 2015, p. 268); "A statement does not have meaning because it is properly constructed; it has meaning because it can be verified. Consequently, any proposition is properly constructed." (Wittgenstein, 2015, pp. 268-269); "No. Because if there is a method of verifying a proposition, then the proposition has meaning. And conversely, if we know how a proposition is verified, then it certainly has meaning." (Wittgenstein, 2015, p. 273)

Wittgenstein opposed the existence of "*a true but unprovable proposition*", considering it a contradictory expression, and at the same time discussed what is "*true*" and what is "*provable*". He explained it with the P-proposition he constructed, "I have constructed a proposition (I will use 'P' to designate it) in Russell's symbolism, and by means of certain definitions and transformations it can be so interpreted that it says: '*P is not provable in Russell's system*'. Must I not say that this proposition on the one hand is true, and on the other hand is unprovable? For suppose it were false; then it is true that it is provable. And that surely cannot be! And if it is proved, then it is proved that it is not provable. Thus it can only be true, but unprovable." (Wittgenstein, 2014, p. 50e) When asked what a true proposition is, he pressed on, "What is called a true proposition in Russell's system, then?" He replied, "For what does a proposition's '*being true*' mean? *'P' is true =P.* (That is the answer.) "(Wittgenstein, 2014, p. 50e) He emphasized that both "*true*" and "*provability*" must correspond to a certain system. He believed that a proposition that is "*true*" is "*provable*". He pointed out: "Just as we ask: '*"provable" in what system*?', so we must also ask: '*"true" in what system*?' '*True in Russell's system*' means, as was said: proved in Russell's system; and '*false in Russell's system*' means: the opposite has been proved in Russell's system." (Wittgenstein, 2014, p. 51e); "Therefore it is not enough to say that

P is verifiable; rather we have to say that P can be verified in a particular system." (Wittgenstein, 2003(PR), p. 170)

Wittgenstein believed that the "*true but unprovable proposition*" is a contradiction and an abuse of language by human beings. He pointed out, "Let us suppose I prove the unprovability (in Russell's system) of P; then by this proof I have proved P? Now if this proof were one in Russell's system—I should in that case have proved at once that it belonged and did not belong to Russell's system. ----That is what comes of making up such sentences. ----But there is a contradiction here!—Well, then there is a contradiction here. Does it do any harm here?" (Wittgenstein, 2014, p.51e); "Such a contradiction is of interest only because it has tormented people, and because this shows both how tormenting problems can grow out of language, and what kind of things can torment us." (Wittgenstein, 2014, p. 52e); "For philosophical problems arise when language goes on holiday." (Wittgenstein, 2013b, p. 37)

Wittgenstein did not agree with Gödel's proof that relied on the interpretability of "*metamathematics*" (*the proof of unprovability*), believing that Gödel's proof was not a mathematical proof in the strict sense, and that an algorithmic proof that solely relied on syntactic standards was the criterion for evaluating metamathematics explanations. For this reason, he made a thorough discussion, "A proof of *unprovability* is as it were a geometrical proof; a proof concerning the geometry of proofs. Quite analogous e.g. to a proof that such-and-such a construction is impossible with ruler and compass. Now such a proof contains an element of prediction, a physical element. For in consequence of such a proof we say to a man: '*Don't exert yourself to find a construction (of the trisection of an angle, say) ----it can be proved that it can't be done.*' That is to say: it is essential that *the proof of unprovability* should be capable of being applied in this way. It must----we might say----be a *forcible reason* for giving up the search for a proof (i.e. for a construction of such-and-such a kind)." (Wittgenstein, 2014, p. 52e); "If it is proved, then it is the terminal pattern in *the proof of unprovability*.—If it is unproved, then *what* is to count as a criterion of its truth is not yet *clear*, and---- we can say---- its sense is still veiled." (Wittgenstein, 2014, p. 52e), "---- Or, suppose P has been proved in a direct way—as I should like to put it—and so in that case there follows the proposition '*P is unprovable*', and it must now come out how this interpretation of the symbols of P collides with the fact of the proof, and why it has to be given up here." (Wittgenstein, 2014, p. 52e); "Suppose non-P is directly proved; it is therefore proved that P can be directly proved! So this is once more a question of interpretation----unless we now also have a direct proof of P. If it were like that, well, that is how it would be." (Wittgenstein, 2014, p. 53e)

It is precisely because of the self-referential contradiction of **Gödel sentence** itself that our interpretation of it has become confused, which Wittgenstein did not allow. He believed that this was not a rule of the game and that such a proposition that led to contradiction was not a meaningful proposition. He pointed out, "In mathematics, the result itself is also the standard of correct calculation. Hence, here one cannot think that someone has followed the rule correctly and yet produced a different calculational pattern." (Wittgenstein, 2003, p. 305); "The technique of our use of truth-functions is characterized by the fact that contradictions do not occur. If a contradiction were to occur in the language-game, that technique would be altered, ----and then we should have departed from the idea that a double negation is an affirmation." (Wittgenstein, 2003, p. 305), "I should have to say: Well, this is really no longer a game." (Wittgenstein, 2003, p. 306); "---- But this contradiction is not a significant proposition! Very well, but nor are the tautologies of logic." (Wittgenstein, 2014, p. 178e)

Obviously, the **Gödel sentence** (*u*) was artificially constructed based on those "*translation rules*" (including ***Gödel coding***, ***the principle of representability***, etc.). **But will things constructed by *translation rules* necessarily follow those *translation rules*?** The author of this article will elaborate in detail below: **Gödel sentence** (*u*) is merely a formal construction; it does not follow the principle of representability and does not fall within the domain of predicate $W(x_1, x_2)$, and thus is an invalid formula. At the same time, **the author also proved that Gödel sentence itself is a paradox**. **This conclusion strongly supports Wittgenstein's doubts and claims, that is, Gödel sentence is a meaningless symbol and not a true proposition, just like the "*set*" *T* in Russell's Paradox.** Although it has the form of a set, it cannot be a set.

6. Question-1: The Gödel sentence is a formula constructed by Gödel based on translation rules (including *Gödel coding* and *the principle of representability*). As a self-referential and self-negating formula, it violates the requirement that "*one of the two must be*" and *the principle of representability*. It does not have *recursion* and thus does not fall within the domain of predicate $W(x_1, x_2)$, and is not a definite and effective formula. We are only talking about this so-called Gödel sentence outside the valid domain of the predicate $W(x_1, x_2)$. The contradiction of the proof originates from Gödel sentence itself, which cannot be attributed to the *incompleteness* of the system; thus, the proof is wrong.

We believe that a *self-referential* formula is not a valid formula, its Gödel number does not belong to the valid domain of the relation W(m,n), and it cannot be used to prove GIT. We know that Gödel sentence *u* is the premise of the proof of GIT, and if it is an invalid formula, then the proof of GIT is also invalid.

In formula $A(x_1) = (\forall x_2)\neg W(x_1, x_2)$, x_1 is the term corresponding to the Gödel number of a **definite** formula $B(y)$, and we substitute x_1 into formula $B(y)$ to obtain $B(x_1)$. The significance of the above $wf.A(x_1)$ is that formula $B(x_1)$ cannot be proven in system *N*. Since we do not limit what this formula $B(x_1)$ is, it can be formula $wf.A(x_1)$ itself. In this way, formula $wf.A(x_1)$ buries the seed of the cyclic definition in its definition, that is, formula $A(x_1)$ can be interpreted as follows: **formula $A(x_1)$ cannot be proved in system *N***, thus becoming a *self-referential* formula. **We define arbitrary formulas by using this *unlimited* predicate $W(x_1, x_2)$, which is exactly like the definition of set *T* in Russell's Paradox.** This shows that the predicate $W(x_1, x_2)$ crosses a certain boundary and goes to its opposite.

Indeed, formally, formula $wf.A(x_1)$ or formula *u* is understandable, and they are all *well-formed* formulas. Similar to the definition of *T* in Russell's Paradox, there is no problem in form, but the explanation is ineffective. **Gödel sentence *u* is *undecidable* in system *N*, which shows only that formal system *N*, such as the concept of "*set*", also needs to be limited.**

How to replace formula $B(x_1)$ with formula $A(x_1)$, that is, to achieve the *self-reference* of a formula? Just do the following: Replace the variable x_1 in the right end of formula $A(x_1)$ with $\mathrm{O}^{(p)}$. The details are as follows:

For formula $A(x_1)=(\forall x_2)\neg W(x_1,x_2)$, we replace x_1 at the right end of the above formula with $\mathrm{O}^{(p)}$, and we obtain **Gödel sentence** $u=A(\mathrm{O}^{(p)})=(\forall x_2)\neg W(\mathrm{O}^{(p)},x_2)$.

Obviously, the above formula is a "*special case*" or "*instance*" of formula $A(x_1)=(\forall x_2)\neg W(x_1,x_2)$. If we allow x_1 to take $\mathrm{O}^{(p)}$, then the formula $A(x_1)$ contains the case of "*self-negation*", that is, the resulting **Gödel sentence** is a "*self-negating*" or "*self-referential*" formula, and at the same time, we will arrive at a paradoxical result.

Specific instructions are as follows:

For the formula $u=A(\mathrm{O}^{(p)})=(\forall x_2)\neg W(\mathrm{O}^{(p)},x_2)$, if *u* is provable, that is, $A(\mathrm{O}^{(p)})$ is provable, then $\exists x_2$ makes $W(\mathrm{O}^{(p)},x_2)$ true, which obviously contradicts the formula *u* itself, $(\forall x_2)\neg W(\mathrm{O}^{(p)},x_2)$, and thus *u* is unprovable.

If *u* is unprovable, that is, the formula $(\forall x_2)\neg W(\mathrm{O}^{(p)},x_2)$ is unprovable, then there are two cases:

(1) The negative form $\neg u$ of the formula *u* is a provable theorem, and therefore the formula *u* itself is unprovable, because a false formula must be unprovable. At this point, $\neg u$ is true, that is, formula $\neg(\forall x_2)\neg W(\mathrm{O}^{(p)},x_2)$ is true; therefore, $\exists x_2$ makes $W(\mathrm{O}^{(p)},x_2)$ valid, and then it is concluded that the formula $A(\mathrm{O}^{(p)})$ is provable, that is, *u* is provable, which contradicts the unprovability of *u*. **In this case, the formula *u* is a paradox.**

(2) The second case is that neither formula *u* nor its negative formula can be proved, that is, formula *u* is a meaningless symbol. In this case, discussing formula *u* is meaningless.

Insofar as *u* is unprovable, we have the formula $A(\mathrm{O}^{(p)})$ unprovable, that is, not $\exists x_2$, making $W(\mathrm{O}^{(p)},x_2)$ true. Notice here: Whatever term x_2 is, numeric term or non-numeric term. If x_2 is a non-numeric term, then it must be an invalid formula for formula $W(\mathrm{O}^{(p)},x_2)$; because if $W(\mathrm{O}^{(p)},x_2)$ is true then x_2 must be the numeric term of the proof of formula $A(\mathrm{O}^{(p)})$ and cannot be a non-numeric term. So in the case that *u* is unprovable, we have: no $\exists x_2$ makes

$W(0^{(p)}, x_2)$ true. That is: for $\forall x_2$, formula $W(0^{(p)}, x_2)$ is not true, that is, formula $(\forall x_2)\neg W(0^{(p)}, x_2)$ is true, that is, $A(0^{(p)})$ is true. So this contradicts the *unprovability* of *u*. At this point, formula *u* is still a paradox.

The above shows that we cannot judge whether **Gödel sentence** is provable or not; Gödel sentence either makes no sense on its own, or it keeps itself in an endless loop of *self-denial*, **so Gödel sentence itself is actually a paradox. This clearly contradicts *the principle of representability* and the requirement of "*one of the two must be*".** Under the requirement of "***one of the two must be***", either **Gödel sentence** or its negative formula can be proved; it is impossible for both a formula and its negation to be unprovable, unless the formula *u* does not belong to the valid domain of the predicate $W(x_1, x_2)$.

Therefore, Gödel sentence cannot fall within the valid domain of the predicate $W(x_1, x_2)$; otherwise, the predicate $W(x_1, x_2)$ would violate *the principle of representability* and lose its *recursiveness*, thus losing the foundation of its own existence. We are only talking about a so-called formula outside the valid domain of the predicate $W(x_1, x_2)$. So, if we want the predicate $W(x_1, x_2)$ to always be valid, we must prohibit it from using this *self-negating* formula, that is, prohibit substituting $0^{(p)}$ for the variable x_1. For this would lead to *self-reference*, i.e., *self-negation*, thus breaking the *recursiveness* of the predicate $W(x_1, x_2)$.

In fact, as a *self-negating* formula, **Gödel sentence** $(\forall x_2)\neg W(0^{(p)}, x_2)$, because it violates ***the principle of representability***, we cannot find an arithmetic statement (or meta-arithmetic statement) corresponding to it, because such a correspondence does not exist.

Or, if we believe that there is such a meta-arithmetic statement as "*The formula for Gödel number a is unprovable*". It is clear that this statement must be explained by the predicate $W(x_1, x_2)$; Without the formal system ***N***, it is impossible to define a purely *self-negating* statement in an arithmetic system, that is, **"*this arithmetic statement is not true*" is not an arithmetic statement at all.**

This *self-negating* formula inherently implies a certain cycle. No formula of any cycle can meet the requirement of *recursion*, and thus inevitably violates *the principle of representability*. The arithmetic relation W(*p,q*) corresponding to the predicate $W(x_1, x_2)$ is defined as follows: The relation W(*p,q*) holds if and only if *p* is the Gödel number of the formula $B(y)$ (where y occurs freely in the formula) and *q* is the Gödel number of the proof of the formula $B(0^{(p)})$ in *N*. Thus, the relation W(*p,q*) depends on the formula $B(0^{(p)})$ in the formal system *N* and its proof, and when this formula is composed of the predicate $W(x_1, x_2)$, the situation becomes extremely complicated. In this case, on the one hand, the relation W(*p,q*) corresponds to the predicate $W(x_1, x_2)$, and they determine each other to form a mutually "*representable*" relation. On the other hand, when the formula $B(y)$ is the formula $A(x_1)$ itself, the formula $B(0^{(p)})$ is the **Gödel sentence** $A(0^{(p)})$, which is formed and determined by the predicate $W(x_1, x_2)$, and the formula $A(0^{(p)})$ itself becomes a judgment object of the predicate $W(x_1, x_2)$, thus forming a sort of cycle, that is, a cycle of "***chicken or egg came first***".

It is precisely because of this cycle that we are unable to determine its authenticity within a limited number of steps, thus losing its *recursiveness* and becoming an *undecidable* proposition. This also indicates that under the "*translation rules*" (including **Gödel coding** and **the principle of representability**), we have constructed a formula *u* that does not adhere to the "*translation rules*". **Gödel, however, brought up this flawed formula to make a point, clearly targeting the wrong person. It should be noted that this formula was constructed by Gödel rather than gradually derived based on *calculation rules*（Lampert, 2018）. Therefore, it is inevitable that it does not follow the "*translation rules*".** Perhaps people do not agree with the above view, but there is a concept in mathematics that can be compared, and this concept is the *limit* concept in calculus. The object and result of a *limit* may not belong to the same domain of discourse; for instance, the limit of a sequence of rational numbers can be an irrational number. It is precisely because the cycle of *self-reference* brings about a paradoxical contradiction that **Gödel sentence** violates *the principle of representability* and becomes a logically invalid formula.

Therefore, as a *self-negating* cyclic formula, Gödel sentence must be excluded from the valid domain of predicate $W(x_1, x_2)$, that is, Gödel sentence does not belong to the valid domain of predicate $W(x_1, x_2)$, and cannot be used to prove the GIT. As we know, not all the well-formed formulas of the formal system *N* can be used to represent a ***number-theoretic function***, such as formula $x_1 = x_1 \wedge y \neq y$ is impossible to represent a ***number-theoretic function***; and **Gödel sentence** is such a kind of well-formed formula, so it does not belong to the effective definition domain of predicate $W(x_1, x_2)$.

There are always some *well-formed* formulas that are outside the valid domain of the predicate $W(x_1, x_2)$. Such formulas exist in form but not in interpretation, creating a kind of idling. It can also be said that the predicate $W(x_1, x_2)$ that decides all the *well-formed* formulas does not exist (otherwise it must be restricted). This is exactly the same as the *T* in **Russell's Paradox** (the so-called *set composed of all sets that do not belong to themselves*). That is, we cannot exhaust all the Turing machines, all the sets, and all the *well-formed* formulas.

Under the *self-negating* circular definition, the validity of Gödel sentence *u* does not hold; that is, ***u*** is not a valid formula. A logical invalid formula has no right or wrong points, and it becomes meaningless to take it as an "*undecidable formula*". Thus, the emergence of ***u*** is not due to the incompleteness of the formal system ***N*** but due to the definition of ***u*** itself. **This is precisely the fundamental reason why the great Wittgenstein believed that Gödel sentence was a logical contradiction.**

Since ***u*** is an invalid formula, we believe that any *language L* of the formal system *N* should not include this formula; that is, there is no interpretation of *u* in the model D_N, so there is no so-called ‘***true but unprovable’*** proposition. Otherwise, no suitable model can exist for any language containing *u*, where we must exclude *u* from the normal language or some restriction on our formal language.

Wittgenstein understood *the principle of representability* and the Gödel numbering (a kind of *translation rule*) very well. Since *the principle of representability* and the *translation rule* (i.e., the Gödel numbering) do not change the *recursion* of the original arithmetic proposition, Wittgenstein argues that “Gödel’s proposition, which asserts something about itself, does not *mention* itself.”(Wittgenstein, 2014, p176e), “but we should also have proved a proposition which, by that *translation rule*, can be expressed: this arithmetical proposition (namely ours) is not derivable.” (Wittgenstein, 2014, p177e). That is, according to the *principle of representability*, Gödel sentence cannot talk about or judge itself.

As Wittgenstein said, *tautology* and *contradictory* logical propositions are meaningless, and they have no rich connotations, which means that nothing is said about the world. **Wittgenstein firmly denied the rationality of this Gödel sentence.** He pointed out that, “Here the constructed proposition was simply another way of writing the constructed number; it had the *form* of a proposition but we don’t compare it with other propositions as a sign *saying* this or that, making *sense*.” (Wittgenstein, 2014, p. 177e). Therefore, as Juliet Floyd points out, “Wittgenstein’s underlying view of Gödel’s proof is that it demonstrates the impossibility of a certain construction – like a demonstration that it is impossible to trisect an angle with ruler and compass. The proof then – being a *proof* – contains ‘a physical element’: we will not accept the goal of ‘constructing’ a (formal) proof in *Principia* of a ‘Gödel sentence’, and will warn people against trying to find derivations of these sentences. In fact, we will insist that any purported derivation of such a sentence is not a derivation *in the relevant sense*. “ (Floyd J., 1995)

7. Question-2: In the proof of Gödel’s Incompleteness Theorem (abbreviated as GIT), we discovered a paradox (that is, Gödel sentence itself is a paradox), and the logical approach of its proof is completely different from that of the proof of the Turing Machine Halting Problem and the solution of Russell’s Paradox.

The scientific responses to **Russell’s Paradox** and **Turing Machine Halting Problem** are that humans deny the existence of this contradiction rather than affirming its existence. In contrast, the proof of the GIT recognizes the rationality of this contradiction, namely, the rationality of **Gödel sentence**, rather than denies it.

As above, we know that the existence of the *undecidable formula* ***u*** in the proof of the GIT is no different from the existence of T in Russell’s Paradox. This is also completely similar to the Turing Machine Halting Problem, but we have come to the opposite conclusion. From the solution of the Turing Machine Halting Problem, we conclude that there is no Turing machine to judge *all Turing machines*. From the GIT, we conclude that the form system *N* is incomplete, rather than questioning the rationality of the *domain* of the predicate $W(x_1, x_2)$ that decides *all the well-formed formulas*, that is, questioning the rationality of Gödel sentence itself. **This unrestricted predicate $W(x_1, x_2)$ is similar to the *relationship of belonging* in Russell's Paradox.**

We argue that the domain of the predicate $W(x_1, x_2)$ is limited, just as is our solution to **Russell's Paradox**: The definition of a set must be limited. That is, ***p*** does not belong to a valid domain of the relation W (m, n). This formula ***u*** is the same as the definition of set *T* in Russell's Paradox, which is a circular definition. The essence of the circular definition is that the idea of *actual infinity* works; that is, **we judge the *"things"* that have not been generated as *"things that already exist"*, thus leading to contradictions.**

Below, we compare the formula ***u*** to the set *T* in Russell's Paradox.

We set *M* as the set of all theorems of the formal system ***N***. In the proof of the GIT:

If $\mathbf{u} \in M$, the relationship W(*p,q*) holds. We derive that both $W(0^{(p)}, 0^{(q)})$ and $\neg W(0^{(p)}, 0^{(q)})$ are theorems for formal systems ***N***, which contradicts the consistency of ***N***, and thus $\mathbf{u} \notin M$.

However, if $\mathbf{u} \notin M$, then the relationship W(*p, q*) does not hold; based on the ω - consistency, we conclude that $\neg\mathbf{u} \notin M$. In this case, although for each *q*, there is $\neg W(0^{(p)}, 0^{(q)})$ holds, but we cannot confirm whether $\mathbf{u} \in M$ is true, that is, we cannot confirm whether $(\forall x_2)\neg W(0^{(p)}, x_2) \in M$ is true; it may or may not be true. Under the different models, we will have the different results. Under the Standard Model (namely, the Normal Model) D_N, only the interpretations of 0, $0^{(1)}$, $0^{(2)}$, $0^{(3)}$, $0^{(4)}$, … are included, and there is no interpretation for the other elements. In this case, for each *q*, there is $\neg W(0^{(p)}, 0^{(q)})$ holds, then we have $(\forall x_2)\neg W(0^{(p)}, x_2) \in M$, that is, $\mathbf{u} \in M$ is true.

Therefore, under the Standard Model D_N, we can derive $\mathbf{u} \notin M$ from $\mathbf{u} \in M$; from $\mathbf{u} \notin M$ we can also derive $\mathbf{u} \in M$. This is clearly a paradox.

While in Russell's Paradox, $T = \{x \mid x \notin x\}$, we have:

If $T \in T$, according to the definition rules, ***T*** is its own element, and we obtain $T \notin T$;

If $T \notin T$, then T meets the definition of the element of ***T*** and thus has $T \in T$. This leads to a conflict.

A comparative analysis of these two proofs obviously shows that they are very similar, and the former is not completely equivalent to the latter, but the former contains the possibility of the latter. That is, under the Standard Model D_N, **Gödel sentence** is a typical **Russell's Paradox,** a self-contradiction.

We can clearly analyze this cycle. In **Russell's Paradox**, the treatment is that, as a set, such a ***T*** does not exist. In the proof of the **Turing Machine Halting Problem**, people also did not recognize the existence of the Turing machine H that judges *"all Turing machines"*. We therefore hold that this *undecidable formula* ***u*** constructed in the proof of GIT, as a contradiction, should not exist and should not be admitted. We should reflect on the other possibilities of the problem as we deal with Russell's Paradox, rather than considering the existence of ***u*** as reasonable and thus concluding the formal system *N* as incomplete. In Gödel's view, the rationality of the existence of ***u*** is based on the rationality of its construction; however, similarly, the construction of ***T*** in Russell's Paradox is also reasonable, but we do not admit the rationality of the existence of ***T*****;** instead, we reflect on whether the definition rule of the set itself is reasonable.

Therefore, the GIT proof shows that its logical approach is obviously different from the former two. **We believe that *u* is a formula with invalid semantics and invalid logic, with its correct form and invalid content, similar to T in Russell's Paradox, which cannot be used to prove the GIT.** As Wittgenstein questions the meaning of ***u***; he claims that it is not a meaningful mathematical proposition, that ***u*** is not available in mathematical proofs and calculations and that we should abandon the natural language interpretation of ***u***. Wittgenstein points out, "What mathematical propositions do stand in need of is a clarification of their grammar, just as do those other propositions." (Wittgenstein, 2014, p. 171e). That is, we must regulate and restrict mathematical propositions such as **Russell's Paradox** and **Gödel sentence**.

8. Why Is Gödel Sentence a Russell's Paradox?

We know that naive set theory gives rise to Russell's Paradox. To avoid the paradox, we developed naive set theory into ZFC axiomatic set theory by the method of "adding axioms", thereby avoiding self-judgment and ultimately eliminating Russell's Paradox. However, any formal system can be reduced to set theory. Without the addition of restrictive axioms, such formal systems are bound to encounter the problem of Russell's Paradox, and Gödel sentence is a concrete manifestation of the paradoxes produced by such formal systems.

We believe that the fundamental reason for the emergence of **Russell's Paradox** and **Gödel sentence** lies in adhering to the idea of *actual infinity*. The following is a detailed analysis:

8.1 Analysis of the Philosophical Reasons Behind Russell's Paradox

The basis of the definition of set is a "*relationship of belonging*", which is a "nested form" between things, and also a *structural* or *topological* relationship between things. This relationship or thing itself is some kind of reality or existence (*objective existence* or *logical existence*), and this *existence* becomes the basis on which we define the new set. Although we do not place any restrictions on such objects of existence, the *existence* of objects is necessarily a prerequisite for the existence of a set. This is our philosophical basic understanding of the concept of *set*. Wittgenstein conducted an in-depth reflection on this "*existence*" in his *Philosophical Investigations*, and he pointed out: "A wish seems already to know what will or would satisfy it; a proposition, a thought, what makes it true—even when that thing is not there at all! Whence this determining of what is not yet there? This despotic demand?" (Wittgenstein, 2013b, p. 218)

Russell's Paradox appears in classical set theory, precisely because we human beings violate the prerequisite of "*the existence of objects*", treat non-existing objects as already existing objects, and include them in the scope of the definition of set (that is, the domain of definition), which leads to the logical cycle and produces the problem of "*the origin of nothing*".

That is, the problem of "*chicken or egg came first*" arises. This kind of thought is essentially a kind of thought of ***actual infinity***, which is also the fundamental reason why we question ***the Principle of Comprehension*** in classical set theory. The philosophical thought on which ***the Principle of Comprehension*** (as the basic principle of constructing the set) depends is the thought of *actual infinity*. The judgment of all objects is the judgment of finished infinity, which is a kind of *thought of actual infinity*.

Under *the Principle of Comprehension*, there is no limit to the object domain. It is this unlimited *all objects* (*actual infinity*) that leads to paradox. There are two understandings of this kind of *all*, one representing *existing* (actually a *potential infinity*), the other representing *existing* and *coming* (which is practically an *actual infinity*). Obviously, the emergence of paradox is due to the latter understanding. The previous understanding that the object of judgment is oriented to history rather than to the future is therefore a potential infinity; the latter understanding is for both the history and the future, which is clearly a judgment of *actual infinity*. This involves determining the criteria for an object of judgment, whether it is a judgment of existence or a judgment of an upcoming future.

According to the law of human understanding of the objective world, the unidirectionality of time, and the directionality of judgment, this fact shows that our human thinking can only judge the exact existing object, which is determined by the hierarchy of knowledge, the law of the development of history, and the hierarchical law of the world. This law of knowledge, the unidirectivity of time, and the directivity of judgment are a kind of thought of *potential infinity*.

On the other hand, if we adhere to the latter standard of judgment, that is, future is also included in our vision of judgment, which will inevitably lead to circular judgment, that is, present judgment is the judgment of past and future, and the result of present judgment must belong to the category of future, which leads to circular judgment and the emergence of Russell's Paradox. This is not only a thought of *actual infinity*, but also a practice that violates the principle of "*unidirectional time*", which will inevitably bring confusion to our understanding.

The ZFC axiomatic set theory is proposed to avoid Russell's Paradox. The application of *the Axiom Schema of Separation* and *Axiom of Regularity* in the ZFC axiom system is a limitation on **the *Principle of Comprehension*. The essence is to limit the objects of our judgment to the existing range, that is, to limit the definition of set to the range determined jointly by known objects (given sets) and given property. For example, it does not allow the set of all sets to exist and the set sequence of infinite descending chains, which is exactly the embodiment of *potential infinity*.** The composition of all sets can be traced back to minimal elements (not sets in themselves). **Therefore, the sets formed by the ZFC axiom system are the judgment of history, the judgment of existing objects,** and even the existence of infinite sets, such as the set of natural numbers, are also the judgment of objects already existing. Therefore, its foundation is based on the solid *earth* (the *earth* composed of the real *minimal elements*), so it must fundamentally eliminate the emergence of paradox. The ZFC axiomatic set theory is a set theory of *potential infinity*, which makes up for the deficiency of the naive set theory. On the contrary, the naive set theory based on **the *Principle of Comprehension*** is based on the idea of *actual infinity*, which leads to the emergence of Russell's Paradox. Therefore, the essence of the **Third Mathematical Crisis** lies in our insistence on the idea of *actual infinity*, which is a crisis caused by the idea of *actual infinity*.

8.2 Analysis of the Philosophical Thoughts behind Gödel Sentence

The fundamental reason why Gödel sentence is a paradox lies in the definition of cycle, and the core is that we adhere to the thought of *actual infinity*, which is completely consistent with the cause of Russell's Paradox. The essence of the circular definition is that the idea of *actual infinity* is at work, that is, we judge the "*things*" that have not yet been generated as "***things that already exist***", which leads to contradictions.

We know that formula $B(y)$ is the basis for formula $W(x_1, x_2)$, and that it must be a definite and valid formula, not an invalid formula. This formula is the key to the proof of Gödel's Incompleteness Theorem. Here we must think of formula $B(y)$ as an "***exact object that already exists***" rather than as an "***object to be generated***", otherwise we will be stuck in a loop. The predicate W here is essentially equivalent to the *relationship of belonging* $\in$ in set theory.

In the proof of Gödel's Incompleteness Theorem, on the one hand, formula $B(y)$, as the object judged by formula $W(x_1, x_2)$, should exist before the latter. On the other hand, formula $A(x_1) = (\forall x_2)\neg W(x_1, x_2)$ consists of the predicate $W(x_1, x_2)$, which exists later than the latter. If the formula $B(y)$ is the formula $A(x_1)$ itself, there is a contradiction—the existence of the formula $B(y)$ is both prior to and later than the existence of $W(x_1, x_2)$, resulting in the formula $B(y)$ being a cyclically defined formula. Although we use the *Gödel numbering* in the process of this definition to disguise the Gödel sentence $(\forall x_2)\neg W(0^{(p)}, x_2)$, the nature of its circular definition does not change, thus forming a cycle of *"chicken or egg came first"*. It is this circular definition that leads to the paradox we found above about Gödel sentence, a *self-negating* paradox. It is also formally equivalent to Russell's Paradox: A *self-negating* formula *u* can both derive its own un-establishment from its own establishment and its own establishment from its own un-establishment; obviously, we are caught in this *dead circle*, and thus completely consistent with Russell's Paradox.

Therefore, we believe that the domain of the predicate $W(x_1, x_2)$ is limited, just as we do with the solution to Russell's Paradox: we must limit the definition of the set. That is to say, *p* does not belong to the valid domain of the relation W(m,n), that is, formula $A(x_1)$ cannot be regarded as the judgment object of predicate $W(x_1, x_2)$, and any formula composed of predicate $W(x_1, x_2)$ cannot be regarded as the judgment object of predicate itself, thus completely eliminating the emergence of a class of formulas like Gödel sentence.

The objects dealt with by the relation W(*p,q*) and the predicate $W(x_1, x_2)$ are not the same. The object of the former is a *potential infinity*, while that of the latter is an *actual infinity*. This clearly contradicts the "*expressible relation*" between them. The relation W(*p, q*) holds if and only if

p is the Gödel number of formula $wf.B(x_1)$, where x_1 appears freely in $wf.B(x_1)$, and q is the Gödel number of the proof of $wf.B(\mathrm{O}^{(p)})$ in the formal system *N*. Relation W(*p, q*) is *primitive recursive* in the model D_N (such as the set of natural numbers), which is a kind of ***potential infinity***, while in the formal system *N*, "*all the well-formed formulas*" is a kind of ***actual infinity***. The object that the predicate $W(x_1, x_2)$ is going to process is a "*completed all*" and an *actual infinity*. This leads to $W(x_1, x_2)$ determining a certain kind of *well-formed formula* defined by itself, which leads to contradictions, hence leading to the emergence of the *undecidable well-formed* formula $u = A(\mathrm{O}^{(p)}) = (\forall x_2)\neg W(\mathrm{O}^{(p)}, x_2)$. Wittgenstein explicitly opposed this "*all*" (that is, a kind of *actual infinity*), stating: "A proposition about all propositions, or all functions, is impossible. Generality in mathematics is illustrated by induction." (Wittgenstein, L., 2003(PR), p. 17)

Here, there is a situation completely similar to Russell's Paradox, and Gödel sentence *u* is similar to $T = \{x \mid x \notin x\}$ in Russell's Paradox. The predicate determines a *well-formed* formula composed of itself; the essence lies in that we insist on the idea of *actual infinity*. In the opinion of Wang Hao, a famous mathematical logician, the main reason Wittgenstein opposes Gödel's proof is this *actual infinity*. He pointed out, "Within any fixed and limited range or in some infinite ranges, there is no implicit construction like Gödel's approach, and this possibility is impossible to achieve." (Fan Yuehong, 2017, p. 172), "The *'true but unprovable'* proposition assumes that an infinite set of positive integers or an infinite set of F(1), F(2), etc., is regarded as a completed whole rather than a potential whole" (Fan Yuehong, 2017, p. 172). Therefore, we believe that the basis of the proof of the GIT is a complete idea of *actual infinity*.

Since the *actual infinity* is different from the *potential infinity*, the relation W(*p, q*) and the predicate $W(x_1, x_2)$ must be equivalent, and this requires that the object that the predicate $W(x_1, x_2)$ processes should not be an *actual infinity*. The former is *primitive recursive*, an impossible self-loop, while the object processed by the latter is an *actual infinity*, which must contain its own judgment of itself, and must appear self-loop, thus destroying its own *recursiveness*. Therefore, the Gödel sentence is bound to be excluded from the effective definition domain of the predicate $W(x_1, x_2)$; that is, the Gödel sentence does not belong to the effective definition domain of the predicate $W(x_1, x_2)$, and the Gödel sentence cannot be used to prove the GIT.

9. Summary

As mentioned above, we have questioned the rationality of GIT's proof from both mathematical and philosophical perspectives, which strongly supports Wittgenstein's viewpoint.

On the one hand, the construction of Gödel sentence *u* is exactly the same as that of ***T*** in Russell's Paradox. The construction of ***u***, based on *the principle of representability*, gives rise to the predicate $W(x_1, x_2)$, and the object of $W(x_1, x_2)$ is "*all well-formed formulas*"; in the construction of

T, the objects are "*all sets*". All of them are confronted with the "*actual infinity*", and thus all face their own judgments of themselves, which leads to contradictions.

On the other hand, the solution to Russell's Paradox is to add axioms and develop naive set theory into ZFC axiomatic set theory, thereby avoiding self-judgment. Gödel, however, attributed the contradiction to the "*incompleteness*" of the formal system ***N*** rather than seeking the cause from the formula *u* itself. Gödel coding and *the principle of representability* are two fundamental tools for constructing Gödel sentence. However, the Gödel sentence violates *the principle of representability* and goes beyond the scope of this principle, thus making itself a logically invalid formula. That is why Wittgenstein regarded it as a meaningless symbol. It is precisely this meaningless symbol that leads to a contradiction, that is: from the fact that *u* holds, it can be deduced that *u* does not hold; from the fact that *u* does not hold, it can be deduced that *u* holds. This can only indicate that the Gödel sentence itself is a paradox.

Therefore, we believe that what should be abandoned is the definition and application of Gödel sentence, rather than, as Gödel holds, that the—system is incomplete and the so-called *'true but unprovable'* proposition does not exist. Just as in Russell's Paradox, what we want to limit is the definition of sets, so the set ***that contains all sets that do not belong to themselves*** does not exist. Gödel attributed the contradiction to the incompleteness of the formal system, which is obviously wrong; Wittgenstein's criticism of Gödel's theorem is completely correct.

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