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Toward a Difference Approach: A Discrete Meat Demand System

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Abstract

While the differential approach to economic analysis is useful, the difference approach is indispensable as almost all economic data are discrete, rather than continuous. Thus, we must investigate the integration of the differential with difference approaches. We show a difference quotient corresponding to a differential quotient, which is generally called a derivative, and a partial difference quotient corresponding to a partial differential quotient, which is generally called a partial derivative. From these, the difference approach produces a discrete demand system with logarithmic mean elasticities as parameters that corresponds to a continuous demand system with point elasticities as parameters produced by the differential approach. These systems should satisfy each budget constraint: the former for finite-change variables and the latter for infinitesimal-change variables. Based on these, we consider a discrete meat demand system, apply it to monthly demand for fresh meat in Japan, and estimate it using a weighted RAS method. The estimated demand system has two desirable properties: each estimated demand (*theoretical value*) *of the conditional demand function coincides with each observed demand, and this system satisfies the difference budget constraint.*

Keywords

difference approach, differential approach, Cournot aggregation, Engel aggregation, logarithmic mean, meat demand system, RAS method

1. Introduction

While the differential approach to economic analysis is useful, the difference approach is indispensable as almost all economic data are discrete, rather than continuous. There are some well-known *differential demand systems* and their variants such as the Rotterdam model (see, e.g., Barten, 1964, 1993; Theil, 1965, 1975/76; Deaton & Muellbauer, 1980b (Chap. 3); Neves, 1994; Clements & Gao, 2014, (Appendix)). However, to estimate these systems using actual data, we must approximate some differential values to their corresponding difference values, which creates new problems (see Section 4 and Appendix B). Thus, we must also investigate a difference approach to avoid these problems.

In this study, we derive and estimate a *difference demand system* whose parameters have to possess some properties, which we call a discrete demand system. When we estimate the parameters of each conditional demand function, we assume that our discrete demand system has two specific features: all independent and dependent variables (prices, total expenditures, and demands) are pre-determined (i.e., measurable, observable, or exogenously given), and all parameters (price and income elasticities, and residuals) of the system are post-determined. (As the *i*th demand is not a dependent variable whenever we estimate this demand system, it would be better for us to use another term instead of the function. However, we use the function for convenience). We can regard all pre-determined variables, variables calculated from these, and real numbers (1 and 0 will be used later) as real data (i.e., real-world data).

In response to this assumption, we develop a new method, which we call the weighted RAS (WRAS) method. The WRAS method can estimate the parameters using only two points data (i.e., initial and terminal data). All the parameters that are estimated using our WRAS display the most desirable properties: each theoretical demand of the conditional demand function calculated using estimated parameters and independent variables coincides with its real value (observable demand), and these parameters satisfy the budget conditions. Thus, we can say that the parameters are consistent with real data (see Sections 5 and 6).

In the below, all variables are assumed to be economic data, and thus they are positive and discrete except for events that assume differentiability for their description. (Wherever we have to consider these events, all variables are assumed to be continuous and these functions are either differentiable or totally differentiable). For simplicity, they are usually not unity when we need to take their logarithms and only the natural logarithm is used. Any economic datum x, which may be called a variable, is commonly given as x_t at point t (e.g., day, month, year). Then, we have two differences such as $\Delta x \equiv x_1 - x_0$ and $\Delta \log x \equiv \log x_1 - \log x_0 = \log(x_1/x_0)$, where the subscript 0 represents an initial point and 1 represents a terminal point. The two differences are always those at these points, except in Subsection 4.3, and Sections 5 and 6. These two differences are also assumed to be non-zero to obtain interesting results, unless we set $\Delta x = 0$ to define a partial difference quotient. Naturally, we consider the limit: $\lim \Delta x \to 0$.

For the differential approach, the two above-mentioned differences (i.e., the differentials) are ordinarily written as dx and $d\log x$. As these differentials cannot be observed or measured, they cannot be used as real data (note that a derivative or a partial derivative is theoretically given by its definition). Thus, there is no differential demand system of which parameters are consistent with real data.

For the difference approach, a logarithmic mean (hereafter log-mean) is essential. The log-mean is defined as

$$L(x) \equiv \frac{\Delta x}{\Delta \log x} = \frac{x_1 - x_0}{\log x_1 - \log x_0} = \frac{x_1 - x_0}{\log(x_1/x_0)} = \frac{x_0 - x_1}{\log(x_0/x_1)}. \quad (1)$$

Sometimes, we write this as $L(x_1, x_0)$ to express two arguments clearly. The log-mean has many useful properties (see, e.g., Carlson, 1972; Stolarsky, 1975; Vartia, 1976; Balk, 2002-3, 2008; Tsuchida, 2014, 2015, 2018). It is always positive and has the following limit:

$$\lim_{\Delta x \to 0} L(x) = x_1 = x_0. \quad (2)$$

Letting $h = x_1/x_0$ and using L'Hopital's rule, we can easily prove Eq. (2) as follows:

$$\lim_{\Delta x \to 0} L(x) = \lim_{h \to 1} \frac{x_0(h-1)}{\log h} = x_0 = x_1.$$

However, as our data are discrete, further proof is desirable (see Appendix D). This limit is used to explain a correspondence between infinitesimal-change and finite-change variables. Likewise, if x_1/x_0 is close to 1, the log-mean can be approximated by three means: arithmetic, geometric, and harmonic (Tsuchida, 2018). Furthermore, it has the following property:

$$L(cx) = \frac{cx_1 - cx_0}{\log(cx_1/cx_0)} = \frac{c(x_1 - x_0)}{\log x_1 - \log x_0} = cL(x), \text{for a positive constant } c. \quad (3)$$

Here, we show the most useful correspondences between the differential and difference approaches (Tsuchida, 2018). From the definition (1), we obtain

$$\Delta \log x = \Delta x / L(x). \quad (4)$$

In contrast, we know the familiar relationship:

$$d \log x = dx/x. \quad (5)$$

From Eqs. (4) and (5), we can find the following correspondences:

$$\text{infinitesimal} - \text{change variables} \left\{ \begin{matrix} d\log x \leftrightarrow \Delta \log x \\ dx \leftrightarrow \Delta x \\ x \leftrightarrow L(x) \end{matrix} \right\} \text{finite} - \text{change variables}, \quad (6)$$

wherein the relationship "A ↔ B" indicates that A corresponds with B, and vice versa. The last correspondence stems from Eq. (2). If $\Delta x \to dx \to 0$, then $\Delta \log x \to d \log x$ and $L(x) \to x (x_1 = x_0)$. The correspondences (6) only hold for $x > 0$ and $x_t > 0$ ($t = 0, 1$) and serve as a bridge between the differential and difference approaches.

The remainder of this paper is organized as follows. In Section 2, we discuss an elasticity of a function using the differential and difference approaches. Our difference approach can produce the difference versions, which are called the log-mean elasticities, corresponding to the point elasticities produced by the differential approach. In there, we first discuss a *difference quotient* corresponding to a differential quotient, which is generally called a derivative. Next, we discuss a *partial difference quotient* corresponding to a partial differential quotient, which is generally called a partial derivative. The partial difference quotient and the log-mean elasticity produced by the difference approach are the most significant concepts introduced in this study, and are the key elements in the derivation of a new demand function. In Section 3, we discuss two specific demand functions and their elasticities. We show that the differential and difference approaches can lead to continuous and discrete log-change demand functions, respectively. Based on these results, in Section 4, we define continuous and discrete demand systems that have to satisfy some conditions from each budget constraint: the former for infinitesimal-change variables and the latter for finite-change variables. While the former conditions are well-known, the latter conditions have never been known. A specific demand system embeds the budget constraint within itself. Hence, the differential budget constraint and difference budget constrain are embedded in the differential and difference versions of the demand system, respectively. In Section 5, we apply our discrete meat demand system to monthly demand for fresh meat in Japan, and the parameters are estimated using the WRAS method. The WRAS method can derive the discrete meat demand system whose parameters are consistent with real data. In Section 6, we present some concluding remarks, wherein we illustrate the consistency with real data using matrix algebra again.

2. Elasticities: Derivations and Definitions

We begin to represent the elasticities produced by the differential and difference approaches using various functions. Our difference approach can derive the difference versions corresponding to the elasticities produced by the differential approach. We use terms such as differential quotient, which is generally called a derivative, in Subsection 2.1 (see also Tsuchida, 2018) and partial differential quotient, which is generally called a partial derivative, in Subsection 2.2 (see, e.g., Takayama, 1974; Berck & Sydsæter, 1991; Bronshtein et al., 2015). In this section and the next two sections, first we explain the differential approach under the expression [Inf-Change], and then we explain the difference approach under the expression [Fin-Change]. When two functions are written on the same line, the first is for the differential approach and the second is for the difference approach in which the subscript t represents a point.

2.1 Functions of One Variable: A Differential Quotient and a Difference Quotient

1) $Y = X^a$ and $Y_t = (X_t)^a$ (a is a constant)

[Inf-Change] The differential approach leads to: $dY = aX^{a-1}dX$. From this, the differential quotient is obtained as follows:

$$dY/dX = aX^{a-1} = a(X^a/X) = a(Y/X).$$

A point elasticity e^* is given by

$$e^* \equiv \frac{XdY}{YdX} = \frac{d\log Y}{d\log X} = a. \tag{7}$$

[Fin-Change] The difference approach leads to: $\Delta \log Y = a\Delta \log X, \Delta Y/L(Y) = a\,\Delta X/L(X)$. Thus, the *difference quotient* is

$$\Delta Y/\Delta X = aL(Y)/L(X).$$

An arc elasticity is defined as

$$[\varepsilon^*] \equiv \frac{A(X)\Delta Y}{A(Y)\Delta X} = \frac{aA(X)L(Y)}{A(Y)L(X)}, \tag{8}$$

wherein $A(x) = (x_1 + x_0)/2$ is the arithmetic mean. Our difference approach induces a new elasticity, called a log-mean elasticity, which is defined as

$$\varepsilon^* \equiv \frac{L(X)\Delta Y}{L(Y)\Delta X} = \frac{\Delta \log Y}{\Delta \log X} = a. \tag{9}$$

If $A(x) \approx L(x)$ (x is X and Y) is assumed, the arc elasticity (8) approaches the log-mean elasticity (9). Comparing Eq. (7) with Eq. (9), we can find close correspondences between the differential and difference approaches. We emphasize these as follows:

$$\frac{dY}{dX} \leftrightarrow \frac{\Delta Y}{\Delta X}, \tag{10}$$

$$e^* = \frac{XdY}{YdX} = \frac{d\log Y}{d\log X} \leftrightarrow \frac{\Delta \log Y}{\Delta \log X} = \frac{L(X)\Delta Y}{L(Y)\Delta X} = \varepsilon^*. \tag{11}$$

The correspondence shown in (10) is easily understood from the following definition of a derivative:

$$\lim_{\Delta X \to 0} \frac{\Delta Y}{\Delta X} = \frac{dY}{dX}.$$

If $\Delta X \to d\mathrm{X} \to 0$, we can employ the above correspondences. Similar correspondences will be found below.

It must be noted that the point and log-mean elasticities usually depend on a point and two points (i.e., an initial point and a terminal point), respectively. Thus, the point elasticity is point-dependent and the log-mean elasticity is two-points-dependent (hereafter twop-dependent). See Subsection 2.2.

2) $Y = cX$ and $Y_t = cX_t$ (c is a positive constant)

[Inf-Change] $dY = cdX$. The differential quotient and point elasticity are

$$\frac{dY}{dX} = c \text{ and } e^* = \frac{XdY}{YdX} = \frac{cX}{Y} = 1.$$

[Fin-Change] $\Delta \log Y = \Delta \log X$, $\Delta Y/L(Y) = \Delta X/L(X)$. The difference quotient is

$$\Delta Y/\Delta X = L(Y)/L(X) = c,$$

wherein $L(Y) = L(cX) = \mathrm{c}L(X)$ from Eq. (3). The arc and log-mean elasticities are

$$[\varepsilon^*] = \frac{A(X)\Delta Y}{A(Y)\Delta X} = \frac{cA(X)}{A(Y)} = 1 \text{ and } \epsilon^* = \frac{L(X)\Delta Y}{L(Y)\Delta X} = 1.$$

Note that $A(Y) = A(cX) = cA(X)$ for a constant c.

2.2 Functions of Two Variables: A Partial Differential Quotient and a Partial Difference Quotient

1) $Y = X^a Z^b$ and $Y_t = (X_t)^a (Z_t)^b$ (a double logarithmic fuction; and a and b are constants)

[Inf-Change] A partial derivative of Y with respect to X is defined as

$$\frac{\partial Y}{\partial X} \equiv \lim_{\Delta X \to 0} \frac{(X+\Delta X)^a Z^b - X^a Z^b}{\Delta X}$$
$$= \lim_{\Delta X \to 0} \frac{(X^a + aX^{a-1}\Delta X + (a(a-1)/2!)X^{a-2}(\Delta X)^2 + \cdots)Z^b - X^a Z^b}{\Delta X} = aX^{a-1}Z^b, \tag{12}$$

which is the function obtained by differentiating Y with respect to X, treating Z as a constant. The total differential of this function is

$$dY = aX^{a-1}Z^{b}dX + bX^{a}Z^{b-1}dZ. \tag{13}$$

From (13), we obtain the following:

$$\frac{\partial Y}{\partial X} = \frac{dY}{dX}|(dZ = 0) = aX^{a-1}Z^{b}, \tag{14}$$

which is a differential quotient dY/dX under the condition: $dZ = 0$ (Note 1) and essentially equivalent to Eq. (12). We designate the middle term in Eq. (14) the partial differential quotient of Y with respect to X. The partial differential quotient of Y with respect to Z is defined in a similar manner.

Thus, the point elasticity of Y with respect to X, e_X, is

$$e_X \equiv \frac{X\partial Y}{Y\partial X} = \frac{aX^{a}Z^{b}}{Y} = a. \tag{15}$$

The point elasticity with respect to Z is similarly defined. These elasticities for the double-log (double logarithmic) function become independent of the point.

[Fin-Change] The difference approach leads to

$$\Delta \log Y \ = a\,\Delta\log X + b\Delta \log Z, \Delta Y = a(L(Y)/L(X))\,\Delta X + b(L(Y)/L(Z))\Delta Z.$$

Our partial difference quotient of Y with respect X is defined as

$$\frac{\delta Y}{\delta X} \equiv \frac{\Delta Y}{\Delta X}|(\Delta Z = 0) = \frac{aL(Y)}{L(X)}, \tag{16}$$

which is the difference quotient $\Delta Y/\Delta X$ under the condition: $\Delta Z = 0$ (Note 2) and corresponds to Eq. (14). The partial difference quotient of Y with respect to Z is defined in a similar manner.

Our log-mean elasticity of Y with respect to X, ε_X, is

$$\varepsilon_X \equiv \frac{L(X)\delta Y}{L(Y)\delta X} = a. \tag{17}$$

While the log-mean elasticity is usually dependent on two points, this elasticity is not.

The partial difference quotient and log-mean elasticity are very effective concepts and play key roles in our discrete demand system. This elasticity also corresponds to Eq. (15). These correspondences are shown as follows, and will be utilized latter if $\Delta X \to d\mathrm{X} \to 0$ and $\Delta \mathrm{Z} = d\mathrm{Z} = 0$:

$$\frac{\partial Y}{\partial X} \leftrightarrow \frac{\delta Y}{\delta X}, \tag{18}$$

$$e_X \equiv \frac{X\partial Y}{Y\partial X} = a \leftrightarrow a = \frac{L(X)\delta Y}{L(Y)\delta X} \equiv \varepsilon_X. \tag{19}$$

The two (differential and difference) approaches to the double-log function produce the same elasticity, that is, the point elasticities at all points between the initial and terminal points equal the log-mean elasticity. (As explained above, the point and log-mean elasticities for other functions are usually point-dependent and twop-dependent, respectively). The log-mean elasticity of Y with respect to Z is analogously defined.

2) $Y = c_1X^{a}Z^{b} + c_2\mathrm{X}$ and $Y_t = c_1(X_t)^{a}(Z_t)^{b} + c_2X_t$

(c_1 and c_2 are positive constants; and a and b are constants)

[Inf-Change] $dY = ac_1X^{a-1}Z^b dX + bc_1X^aZ^{b-1}dZ + c_2dX$. The partial differential quotient of *Y* with respect to *X* is

$$\frac{\partial Y}{\partial X} = \frac{dY}{dX}|(dZ = 0) = ac_1X^{a-1}Z^b + c_2,$$

Thus, the point elasticity of *Y* with respect to *X*, e_X, is

$$e_X = \frac{X\partial Y}{Y\partial X} = \frac{ac_1X^aZ^b + c_2X}{Y}. \quad (20)$$

This point elasticity is dependent on a specific point because the variables *X*, *Z*, and *Y* are those values at that point. The point elasticity of *Y* with respect to *Z* is analogously defined.

[Fin-Change] Letting $G_t = c_1(X_t)^a(Z_t)^b$ and $F_t = c_2X_t$, the difference approach gains the following (see Tsuchida, 2018):

$$\Delta \log G = a\Delta \log X + b\,\Delta\log Z\,, \Delta G = a(L(G)/L(X))\Delta X + b(L(G)/L(Z))\Delta Z,$$

$$\Delta F = c_2\Delta X, \Delta Y = \Delta G + \Delta F,$$

$$\therefore\ \Delta Y = (a(L(G)/L(X)) + c_2)\Delta X + b(L(G)/L(Z))\Delta Z$$

Thus, the partial difference quotient of *Y* with respect to *X* is

$$\frac{\delta Y}{\delta X} = \frac{\Delta Y}{\Delta X}|(\Delta Z = 0) = \frac{aL(G)}{L(X)} + c_2,$$

wherein $L(G) = c_1L(X^aZ^b)$. The log-mean elasticity of *Y* with respect to *X*, ε_X, is

$$\varepsilon_X = \frac{L(X)\delta Y}{L(Y)\delta X} = \frac{ac_1L(X^aZ^b) + c_2L(X)}{L(Y)}. \quad (21)$$

This log-mean elasticity is dependent on two points. We can see that Eq. (21) corresponds to Eq. (20). The log-mean elasticity of *Y* with respect to *Z* is similarly defined.

3. Continuous and Discrete Log-Change Demand Functions and Their Elasticities

In this section, we discuss two specific demand functions and their elasticities. We also use two approaches: differential and difference approaches. To simplify our equations, there are two commodities, *i* and *j*. Our fundamental *i*th demand functions are

$$q_i = f_i\left(p_i, p_j, y\right) \text{and } q_{ti} = f_{ti}\left(p_{ti}, p_{tj}, y_t\right),$$

wherein *p*, *q*, and *y* represent price, quantity, and income (i.e., total expenditure), respectively, and the subscripts *i* and *j* represent the commodities. In this section, the budget constraints are discarded.

1) The *i*th demand functions: $q_i = cp_i^{a_{ii}}p_j^{a_{ij}}y^{bi}$ and $q_{ti} = c(p_{ti})^{a_{ii}}(p_{tj})^{a_{ij}}(y_t)^{bi}$

(a double logarithmic fuction, c is a positive constant; and a_{ii}, a_{ij}, and b_i are constants)

[Inf-Change] Taking the logarithms of both sides yields:

$$\log q_i = \log c + a_{ii}\log p_i + a_{ij}\log p_j + b_i \log y.$$

The differential approach produces the following log-change demand function and elasticities:

$$d\log q_i = a_{ii} d\log p_i + a_{ij} d\log p_j + b_i d\log y,$$

$$e_{ij} \equiv \frac{p_j}{q_i}\frac{\partial q_i}{\partial p_j} = a_{ij}\ (j = i \text{ and } j) \text{ and } h_i \equiv \frac{y}{q_i}\frac{\partial q_i}{\partial y} = b_i,$$

in which e_{ij} is the point price elasticity of q_i with respect to price p_j and h_i is the point income elasticity of q_i with respect to income y. For simplicity, we call these the point elasticities. Thus, the log-change demand function can be rewritten as

$$d\log q_i = e_{ii} d\log p_i + e_{ij} d\log p_j + h_i d\log y. \quad (22)$$

We call this the continuous log-change demand function. We use *continuous* and *discrete* to stress the demand function or system applied to continuous and discrete data, respectively.

[Fin-Change] The difference approach leads to the following log-change demand function and elasticities:

$$\Delta\log q_i = a_{ii}\Delta\log p_i + a_{ij}\Delta\log p_j + b_i\Delta\log y,$$

$$\varepsilon_{ij} \equiv \frac{L(p_j)}{L(q_i)}\frac{\delta q_i}{\delta p_j} = a_{ij}\ (j = i \text{ and } j) \text{ and } \eta_i \equiv \frac{L(y)}{L(q_i)}\frac{\delta q_i}{\delta y} = b_i,$$

in which ε_{ij} is the log-mean price elasticity of q_i with respect to p_j and η_i is the log-mean income elasticity of q_i with respect to y. We also call these the log-mean elasticities. The demand function, which is called the discrete log-change demand function, can be rewritten as

$$\Delta\log q_i = \varepsilon_{ii}\Delta\log p_i + \varepsilon_{ij}\Delta\log p_j + \eta_i\Delta\log y. \quad (23)$$

2) The *i*th demand functions: $q_i = \mathrm{c} + a_{ii}p_i + a_{ij}p_j + b_i\mathrm{y}$ and $q_{ti} = c + a_{ii}p_{ti} + a_{ij}p_{tj} + b_i y_t$

$(c, a_{ii}, a_{ij}, \text{and } b_i \text{are constants})$

[Inf-Change] The differential approach leads to: $dq_i = a_{ii}dp_i + a_{ij}dp_j + b_i dy$. Thus, we have:

$$\mathrm{d}\log q_i = \frac{a_{ii}p_i}{q_i}\mathrm{d}\log p_i + \frac{a_{ij}p_j}{q_i}\mathrm{d}\log p_j + \frac{b_i y}{q_i}\mathrm{d}\log y.$$

The point elasticities are

$$e_{ij} = \frac{a_{ij}p_j}{q_i}\ (j = i \text{ and } j) \text{and} h_i = \frac{b_i y}{q_i}.$$

Therefore, the continuous log-change demand function for this case can also be given by (22).

[Fin-Change] The difference approach leads to: $\Delta q_i = a_{ii}\Delta p_i + a_{ij}\Delta p_j + b_i\Delta y$. This produces

$$\Delta\log q_i = \frac{a_{ii}L(p_i)}{L(q_i)}\Delta\log p_i + \frac{a_{ij}L(p_j)}{L(q_i)}\Delta\log p_j + \frac{b_i L(y)}{L(q_i)}\Delta\log y.$$

The log-mean elasticities are

$$\varepsilon_{ij} = \frac{a_{ij}L(p_j)}{L(q_i)}\ (j = i \text{ and } j) \text{ and } \eta_i = \frac{b_i L(y)}{L(q_i)}.$$

Hence, the discrete log-change demand function for this case can also be given by (23).

4. Continuous and Discrete Demand Systems and the Aggregations of Their Elasticities

4.1 Continuous Demand System and Discrete Demand System

From the explanations in Section 3, it is inferred that we can derive a general *differential* demand system from Eq. (22) and a *difference* demand system from Eq. (23). In this section, we provide more details about these systems.

[Inf-Change] The pedantic derivation of the general differential case is easy. Our fundamental demand system is:

$$q_i = q_i(\mathbf{p}, y)(i = 1, 2, \dots, n),$$

Where $\mathbf{p} = \{p_1, p_2, \dots, p_n\}$, $y = \Sigma p_i q_i$, and n are the price vector, income (total expenditure), and the number of all commodities in this system, respectively. The summation $\sum_i x_i$ or $\sum x_i$ is always made over all values of i. Here q_i is implicitly derived using constrained utility maximization. The total differentiation of this function yields:

$$dq_i = \sum_j \frac{\partial q_i}{\partial p_j} dp_j + \frac{\partial q_i}{\partial y} dy.$$

Thus, we have the general differential demand system as follows:

$$d \log q_i = \sum_j e_{ij} d \log p_j + h_i d \log y \quad (i = 1, 2, \dots, n). \tag{24}$$

Recall that e_{ij} denotes the point price elasticity and h_i denotes the point income elasticity. These elasticities possess various properties produced using a utility function and should satisfy the two conditions derived by the differential budget constraint (30) below (see also Subsection 4.3 and Appendix A). If we substitute the Slutsky equation and another budget constraint (see Appendix B) into (24), we obtain a model similar to the Rotterdam model or its variant (see, e.g., Barten, 1993; Matsuda, 2005; Clements & Gao, 2014). We do not employ the Slutsky equation, so our price elasticities are always Marshallian.

We define a continuous demand system as shown in Eq. (25), based on Eq. (24):

$$d \log q_i = \sum_j e_{ij} d \log p_j + h_i d \log y + u_i \ \ (i = 1, 2, \dots, n), \tag{25}$$

where the final term u_i is the residual and exhibits all of the effects induced by the other factors that are not employed as explanatory variables (e.g., weather and consumer sentiment). The elasticities and residuals should satisfy the above conditions and the residual condition that is explained later, respectively.

[Fin-Change] The pedantic derivation of a general difference demand system is impracticable, because the difference approach is only applied to a specific demand system. Considering the derivation process of Eq. (23) and the correspondences between the differential and difference approaches in (6), (10), (11), (18), and (19), we define a difference demand system:

$$\Delta \log q_i = \sum_j \varepsilon_{ij} \Delta \log p_j + \eta_i \Delta \log y \quad (i = 1, 2, \ldots, n), \tag{26}$$

wherein ε_{ij} and η_i are the log-mean price and income elasticities that should satisfy the two conditions derived using the difference budget constraint discussed below. If we consider numerous demand systems based on microeconomic theory, we can derive this system from most of them. For example, we can derive such a system from the linear expenditure system (LES) as shown in Appendix A (Note 3). Recall that the LES is far removed from Eq. (26). See also Appendix C. If we assume a certain demand system derived using constrained utility maximization, the characteristics of the utility function regulate these elasticities. If $\Delta p_i \to dp_i \to 0$ and $\Delta q_i \to dq_i \to 0$ (for all i); and $\Delta y \to dy \to 0$, the difference demand system (26) approaches the general differential demand system (24).

We use this to define a discrete demand system:

$$\Delta \log q_i = \sum_j \varepsilon_{ij} \Delta \log p_j + \eta_i \Delta \log y + \mu_i \quad (i = 1, 2, \ldots, n), \tag{27}$$

where the final term μ_i is the residual corresponding to u_i above. The conditions that their elasticities should satisfy are the same as those in Eq. (26) and the condition regarding the residual is discussed below.

We may derive the following approximation formed from Eqs. (24) and (26):

$$\Delta \log q_i \approx \sum_j e_{ij} \Delta \log p_j + h_i \Delta \log y \; (i = 1, 2, \ldots, n). \tag{28}$$

This log-change demand system is similar to the Rotterdam model or its variants (e.g., Barten, 1964; Theil, 1965, 1975/76; Clements & Gao, 2014). If we use log-change values $\Delta \log q_i$, $\Delta \log p_i$, and $\Delta \log y$ as in (28), we have to employ the budget constraint for finite-change variables shown below. Therefore, the approximation in (28) may not satisfy this difference budget constraint because this constraint only applies to the log-mean elasticities. See Appendix B for further details.

4.2 Conditions for Elasticities and Residuals

Frisch (1959) showed the conditions that the point elasticities of a demand system have to satisfy (see also Deaton & Muellbauer, 1980b; Barten, 1993). We follow some of these conditions and derive some new conditions for log-mean elasticities.

4.2.1 Homogeneity Conditions Derived from the Homogeneous Function

The homogeneity condition may be meaningful for infinitesimal-change variables, but not necessarily meaningful for finite-change variables.

[Inf-Change] The demand function should be homogeneous of degree zero in prices and income. Thus, we have

$$q_i = q_i(\mathbf{p}, y) = q_i(\gamma \mathbf{p}, \gamma y), \; (i = 1, 2, \ldots, n),$$

wherein γ is a positive constant. This is known as the homogeneity restriction. Since we have the following from Euler's theorem:

$$0 = \sum_j p_j \frac{\partial q_i}{\partial p_j} + y \frac{\partial q_i}{\partial y}, (i = 1, 2, \dots, n),$$

thus, the homogeneity condition for the point elasticities is given by

$$0 = \sum_j e_{ij} + h_i, (i = 1, 2, \dots, n). \quad (29)$$

[Fin-Change] If the demand function is homogeneous of degree zero, we have the following formal equation:

$$q_{1i} = q_{1i}(\mathbf{p_1}, y_1) = q_{0i}(\mathbf{p_0}, y_0) = q_{0i}, (i = 1, 2, \dots, n),$$

wherein $\mathbf{p_1} = \gamma \mathbf{p_0}$, $\mathbf{p_0} = \{p_{0i}\}$ (price vector), $y_1 = \gamma y_0$, $y_0 = \Sigma p_{0i} q_{0i}$, and γ is a positive constant. We also call this the homogeneity restriction. As the restriction requires the differences of the demands to become null (i. e., $\Delta q_i = 0$) for all commodities, our difference quotient and partial difference quotient are inactive. Therefore, we cannot define the homogeneity condition for the log-mean elasticities.

4.2.2 Conditions Derived from the Budget Constraints

The budget constraints are as follows:

$$y = \sum_i p_i q_i, \text{for infinitesimal} - \text{change variables},$$

$$y_t = \sum_i m_{ti}, m_{ti} = p_{ti} q_{ti} \text{ is the expenditure for finite} - \text{change variables},$$

$$1 = \sum_i w_{ti}, \ w_{ti} = p_{ti} q_{ti} / y_t \text{ is the budget share for finite} - \text{change variables}.$$

For finite-change variables, our difference approach must utilize the transformation m_{ti} or w_{ti} as defined above (see also Tsuchida, 2018). We call the former Transformation-M and the latter Transformation-W.

[Inf-Change] $dy = \sum_i (q_i dp_i + p_i dq_i)$

$$\frac{dy}{y} = \sum_i \left(\frac{q_i p_i}{y} \frac{dp_i}{p_i} + \frac{p_i q_i}{y} \frac{dq_i}{q_i} \right) = \sum_i w_i \left(\frac{dp_i}{p_i} + \frac{dq_i}{q_i} \right),$$

$$\therefore d \log y = \sum_i w_i (d \log p_i + d \log q_i). \quad (30)$$

Equation (30) is the differential budget constraint, from which we produce two conditions: the Engel condition on the income elasticities and Cournot conditions on the price elasticities (Note 4):

$$\sum_i w_i h_i = 1 (i = 1, 2, \dots, n), \text{ (Engel condition)};$$

$$\sum_i w_i e_{ij} = -w_j (i \text{ and } j = 1, 2, \dots, n), \text{ (Cournot conditions)}.$$

These conditions hold only for the point elasticities. The general differential demand system (24) should satisfy these conditions. The important point here is that the differential demand system based on a specific demand system automatically satisfies these conditions because the budget constraint (30) is embedded within itself (see Subsection 4.3 and Appendix A). The same is true for the difference case presented below.

As a continuous demand system needs a condition regarding the residuals, we derive this condition. Multiplying both sides of Eq. (25) by w_i and summing for i, we get:

$$\sum_i w_i d\log q_i = \sum_i w_i \left(\sum_j e_{ij}\, d\log p_j + h_i d\log y + u_i\right).$$

Using the two conditions and the budget constraint (30), we have the following Residual condition:

$$\sum_i w_i u_i = 0.$$

Theil's approximation (Theil, 1975/76, Eq. (2.4) in Chap. 2) to the budget constraint (30) is

$$\Delta\log y \approx \sum_i A(w_i)\,(\Delta\log p_i + \Delta\log q_i),$$

wherein $A(w_i)$ = $(w_{1i} + w_{0i})/2$ is the arithmetic mean of the two budget shares. This approximation produces different conditions to those outlined above. For a more detailed discussion, see Appendix B.

[Fin-Change] Using Transformation-M, we derive two types of conditions: the Engel condition on the income elasticities and Cournot conditions on the price elasticities (Similar conditions derived using Transformation-W are shown in Appendix A). To obtain these conditions, we utilize the following difference budget constraint:

$$\Delta y = \sum_i \Delta m_i,$$

$$\therefore \Delta\log y = \frac{\Delta y}{L(y)} = \sum_i \frac{L(m_i)}{L(y)}\Delta\log m_i = \sum_i \frac{L(m_i)}{L(y)}\ \Delta\log p_i + \sum_i \frac{L(m_i)}{L(y)}\Delta\log q_i. \quad (31)$$

Thus, we have

$$\textstyle\sum_i \big(L(m_i)/L(y)\big)\eta_i = 1\big(i = 1, 2, \dots, n\big), \text{ (Engel condition)}; \quad (32)$$

$$\textstyle\sum_i \big(L(m_i)/L(y)\big)\varepsilon_{ij} = -\big(L(m_j)/L(y)\big)\big(i \text{ and } j = 1, 2, \dots, n\big), \text{ (Cournot conditions)}. \quad (33)$$

We call these conditions the M-Engel and M-Cournot conditions. If the difference demand system (26) is not based on a specific demand system, these conditions must be satisfied.

Furthermore, the residuals of the discrete demand system need a new condition. Multiplying both sides of (27) by $L(m_i)/L(y)$ and summing for i, we have

$$\sum_i \frac{L(m_i)}{L(y)}\Delta\log q_i = \sum_i\sum_j \frac{L(m_i)}{L(y)}\varepsilon_{ij}\Delta\log p_j + \sum_i \frac{L(m_i)}{L(y)}\eta_i\Delta\log y + \sum_i \frac{L(m_i)}{L(y)}\mu_i$$

$$= -\sum_j \frac{L(m_j)}{L(y)}\Delta\log p_j + \Delta\log y + \sum_i \frac{L(m_i)}{L(y)}\mu_i,$$

wherein we use Eqs. (32) and (33). Thus, we produce the M-Residual condition as follows:

$$\sum_i \frac{L(m_i)}{L(y)}\mu_i = 0. \quad (34)$$

Here, we explain the advantage of using the above three conditions as constraints over the parameters (elasticities and residuals). Since our discrete demand system is not derived from a specific demand system, this should satisfy these conditions. Thus, we can exploit these conditions. This demand system is composed of n demand functions, and these parameters are exhibited as an $n \times (n+2)$ matrix, wherein the columns relate to the three conditions and each row relates to each demand function in (27). We employ these features to estimate the parameters. A more detailed explanation is provided in Subsection 5.2 and Section 6.

It is also worth noting that the weights of the aggregation, $L(m_i)/L(y)$, are those of an ideal log-change index. We have known two ideal log-change indices: the Montgomery index and the Vartia-Sato index (Vartia, 1976; Sato, 1976; Balk, 2008; Tsuchida, 2014). From our budget constraint (31), we can identify the Montgomery index. (For the Vartia-Sato index, see Appendix A). The first and second terms on the right-hand side of Eq. (31) are the ideal log-change price and quantity indices, respectively. We call these weights the Montgomery weights, which leads to the well-known inequality:

$$\sum_i L(m_i) \leq L\left(\sum_i m_i\right) = L(y). \tag{35}$$

The ingenious proof of the inequality (35) was discussed by Balk (2008, p. 87), who used Jensen's inequality for a convex (or concave) function. As our data are assumed to be discrete, we should seek further proof, which is presented in Appendix D.

4.3 Elasticities That Satisfy the Budget Conditions

It should be noticed that the point elasticities and log-mean elasticities are usually point-dependent and twop-dependent, respectively. To clearly illustrate this, first we explain the difference case for Transformation-W in detail and then shortly explain the differential case.

To estimate the parameters of each difference demand system (26) using yearly data from 2000 to 2015, we use the difference demand system (36), its budget constraint (37), and its budget conditions (Engel (38) and Cournot (39) conditions) in years t and $s = t - 1$ as follows:

$$\Delta \log q_{tsi} = \sum_j \varepsilon_{ij}^{ts} \Delta \log p_{tsj} + \eta_i^{ts} d \log y_{ts}, \tag{36}$$

$$\Delta \log y_{tsi} = \sum_i \frac{L(w_{ti}, w_{si})}{\sum_k L(w_{tk}, w_{sk})} \Delta \log p_{tsi} + \sum_i \frac{L(w_{ti}, w_{si})}{\sum_k L(w_{tk}, w_{sk})} \Delta \log q_{tsi}. \tag{37}$$

$$\sum_i \left(L(w_{ti}, w_{si}) \Big/ \sum_k L(w_{tk}, w_{sk}) \right) \eta_i^{ts} = 1, \tag{38}$$

$$\sum_i \left(L(w_{ti}, w_{si}) \Big/ \sum_k L(w_{tk}, w_{sk}) \right) \varepsilon_{ij}^{ts} = -\left(L\left(w_{tj}, w_{sj}\right) \Big/ \sum_k L\left(w_{tk}, w_{sk}\right) \right). \tag{39}$$

The superscripts and subscripts t and s represent the year; and subscripts i, j, and k represent commodities. Each difference is expressed as follows: $\Delta \log q_{tsi} = \log q_{ti} - \log q_{si}$, etc. The superscript and subscript t represents the terminal year, whereas s represents the initial year. We find the Vartia-Sato index from the budget constraint (37), and the weights therein are Vartia-Sato's, which sum to 1. This budget constraint produces the two conditions (38) and (39) (see Appendix A). The demand system (36) should satisfy (37). We consider that these equations hold for every pair of years t

(2001, 2002, …, 2015) and *s*, and all *i*, *j*, and *k* (1, 2, …, n). From this, all elasticities are twop-dependent (good examples of the LES are shown in Appendix A).

If all elasticities are independent of all points, we have the following equation (40) instead of (36):

$$\Delta \log q_{tsi} = \sum_j \varepsilon_{ij} \Delta \log p_{tsj} + \eta_i \Delta \log y_{ts}. \tag{40}$$

To satisfy the budget constraint (37) for every pair of years, we have the following for all *t*:

$$\sum_i l_{tsi}\varepsilon_{ij} = -l_{tsj},\ \sum_i l_{tsi}\eta_i = 1 (t = 2001, 2002, \ldots, \text{and all } i \text{ and } j). \tag{41}$$

In (41), we rewrite the Vartia-Sato weight as $l_{tsi} = L(w_{ti}, w_{si})/\sum_k L(w_{tk}, w_{sk})$. From these equations, we can obtain the elasticities to satisfy Eq. (37). Without loss of generality, we assume three commodities (*i*, *j*, and *k*= 1, 2, 3) and any three years (*t* = *r*, *u*, *v*: and *r* ≠ *u*, *u* ≠ *v*, etc.) from 2001 to 2015. First, we solve the Cournot conditions. These are given by:

$$\begin{pmatrix} \varepsilon_{11}, \varepsilon_{21}, \varepsilon_{31} \\ \varepsilon_{12}, \epsilon_{22}, \varepsilon_{32} \\ \varepsilon_{13}, \varepsilon_{23}, \varepsilon_{33} \end{pmatrix} \begin{pmatrix} l_{ts1} \\ l_{ts2} \\ l_{ts3} \end{pmatrix} = \begin{pmatrix} -1, 0, 0 \\ 0, -1, 0 \\ 0, 0, -1 \end{pmatrix} \begin{pmatrix} l_{ts1} \\ l_{ts2} \\ l_{ts3} \end{pmatrix}. \tag{42}$$

We can use Eq. (42) for all vectors $\mathbf{l_{ts}} = (l_{ts1}, l_{ts2}, l_{ts3})$ for *t* = 2001, 2002, …, 2015. Thus, we obtain the price elasticities to satisfy the Cournot conditions as follows:

$$\varepsilon_{ii} = -1, \varepsilon_{ij} = 0\ (j \neq i) \text{ for all } i \text{ and } j.$$

Next, we solve the Engel condition. This is given by

$$\begin{pmatrix} l_{rr-11}, l_{rr-12}, l_{rr-13} \\ l_{uu-11}, l_{uu-12}, l_{uu-13} \\ l_{vv-11}, l_{vv-12}, l_{vv-13} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} l_{rr-11}, l_{rr-12}, l_{rr-13} \\ l_{uu-11}, l_{uu-12}, l_{uu-13} \\ l_{vv-11}, l_{vv-12}, l_{vv-13} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \text{ (Note 5)} \tag{43}$$

From (43), we obtain $\eta_i = 1$ for all *i*.

Substituting these elasticities into Eq. (40), we obtain the following primitive difference demand system:

$$\Delta \log q_{tsi} = -\Delta \log p_{tsi} + \Delta \log y_{ts}, \text{for all } t\ (2001, \ldots, 2015) \text{ and } i.$$

Multiplying both sides of the above equation by $L(w_{ti},w_{si})$ and summing for *i*, we obtain the budget constraint (37). Hence, this system satisfies this constraint.

The general differential demand system (24) should satisfy the differential budget constraint (30) and its budget conditions (the Engel and Cournot conditions) in every year; that is, all elasticities must be point-dependent (see the differential version of the LES shown in Appendix A). If all elasticities are independent of all points, we can obtain these elasticities using similar procedures to the above-mentioned difference case. Therefore, we have the following primitive differential demand system:

$$d \log q_{ti} = -d \log p_{ti} + d \log y_t, \text{for all } t\ (2000, \ldots, 2015) \text{ and all } i,$$

Where $d \log q_{ti}$, $d \log p_{ti}$, and $d \log y_t$ are the differentials in year *t*. This system also satisfies the differential budget constraint (30).

An example of yearly demand system for the above primitive difference and differential demand systems is

$$\log q_{ti} = -\log p_{ti} + \log y_t + \log(1/n), (i = 1, 2, \dots, n).$$

Thus, all budget shares take the same value, that is, $w_{ti} = (p_{ti} q_{ti})/y_t = 1/n$ for all t and i. This demand system is derived from the utility function: $U_t = \sum_i \log q_{ti}$ (see Phlips, 1974, pp. 65-66). (By maximizing the Lagrangean with respect to q_{ti} and minimizing it with respect to a Lagrangean multiplier λ, we have $1/q_{ti} = \lambda p_{ti}$ and $y_t = \sum p_{ti} q_{ti}$. From the two equations, we obtain $\lambda = n/y_t$). Generally, we have 16 sets of point elasticities for the differential demand system and 15 sets of log-mean elasticities for the difference demand system. This means that most of the point elasticities and log-mean elasticities are point-dependent and twop-dependent, respectively (Note 6).

5. Estimation of the Discrete Meat Demand System

5.1 Estimating Method

We showed in Section 4 that the discrete demand system is given by (27) and their parameters (elasticities and residuals) should satisfy the M-Engel, M-Cournot, and M-Residual conditions. In this section, we estimate these elasticities and residuals using monthly expenditure data for fresh (or raw) meat purchases over the previous year in Japan. An explanation of the data used is provided in Appendix E.

Our discrete meat demand system in a given month is

$$\Delta \log q_i = \sum_j \varepsilon_{ij} \Delta \log p_j + \eta_i \Delta \log y + \mu_i \quad (i \text{ and } j = 1, 2, 3, 4). \qquad (44)$$

We do not suppose a specific demand system. Fresh meat is composed of four commodities: beef (i or j = 1), pork (i or j = 2), chicken (i or j = 3), and others (i.e., other meats, i or j = 4). The variables q_i, p_i, and μ_i are, respectively, the ith demand, price, and residual, whereas y is the total expenditure on these types of meat. The residual includes the contributions of other factors excluding p_i (I = 1, 2, 3, 4) and y. Most of those contributions may be induced by substitutes and complements for fresh meat (e.g., ham, sausage, and cooked foods such as croquette and Hamburg steak). Implicitly, assuming Transformation-M, all elasticities in (44) are M-elasticities and should satisfy the three above-mentioned conditions. We call Eq. (44) the M-Demand equation conditions in the below. (When we employ a continuous meat demand system such as that in (25), it worth mentioning the homogeneity condition (29). Each conditional demand function may not satisfy this condition since the prices of the substitutes and complements for fresh meat are not contained in the function.)

We employ a two-stage method to estimate the parameters in (44). In the first stage, we estimate these parameters by Ordinary Least Squares (OLS) using all monthly data. Hence, we cannot consider the difference budget constraint (31). The results provide information about the rough values of the parameters in the next stage, where we regulate these values to satisfy the above-mentioned conditions using the WRAS method. Our main aim in this section is not to evaluate these final results, but rather to explain the new method.

5.2 Estimated Results

[First Stage]

First, we calculated the shares of average monthly expenditure on the various types of fresh meat from 2014 to 2016 as shown in Table 1. It can be seen that in Japan, pork has the largest share, followed by beef, except for December. For December, beef has the largest share, which may stem from a seasonal effect (in particular, the preparation of *sukiyaki* for dinner).

Table 1. Average Expenditure Shares

Month	Beef	Pork	Chicken	Others	Month	Beef	Pork	Chicken	Others
Jan.	0.297	0.418	0.217	0.068	Jul.	0.304	0.424	0.206	0.066
Feb.	0.269	0.438	0.222	0.070	Aug.	0.322	0.414	0.194	0.071
Mar.	0.284	0.428	0.222	0.067	Sep.	0.286	0.423	0.220	0.070
Apr.	0.290	0.425	0.218	0.067	Oct.	0.281	0.428	0.220	0.070
May	0.310	0.412	0.211	0.068	Nov.	0.282	0.424	0.223	0.071
Jun.	0.293	0.428	0.213	0.066	Dec.	0.382	0.344	0.211	0.063

Next, we estimate the parameters of each conditional meat demand function using OLS. We added constant terms to Eq. (44) and deleted the residuals. The data were the following. For example, the *i*th per capita demand in April is

$$\Delta \log q_i = \log q_{ti} - \log q_{t-1i}\, , t = 2016 \text{ and } 2015.$$

We have two log-change values in April (2016/15 and 2015/14). Similarly, $\Delta \log p_i$ are given. Total expenditure in April is given by

$$\Delta \log y = \log y_t - \Delta \log y_{t-1}\, , t = 2016 \text{ and } 2015.$$

Table 2. Estimated Parameters at the First Stage

Demand	The price of Beef $\Delta\log p_1$	Pork $\Delta\log p_2$	Chicken $\Delta\log p_3$	Others $\Delta\log p_4$	The total expenditure $\Delta\log y$	Constant	R^2	Corrected R^2
Beef	**-0.9854**	0.1261	-0.6767	-0.3369	**1.2508**	-0.0444	0.8118	0.7491
(i= 1)	’(-5.19)	’(0.27)	’(-2.38)	’(-1.48)	’(2.83)	’(-2.68)		
Pork	-0.0081	**-0.9978**	0.4039	0.2018	**0.6094**	0.0204	0.7083	0.6111
(i= 2)	’(-0.09)	’(-4.24)	’(2.87)	’(1.82)	’(2.78)	’(2.48)		
Chicken	-0.0212	-0.0428	**-0.7184**	-0.1060	**1.0371**	0.0206	0.5975	0.4633
(i= 3)	’(-0.15)	’(-0.12)	’(-3.28)	’(-0.61)	’(3.04)	’(1.62)		
Others	0.0628	-0.4236	-0.0112	**-0.5429**	**1.4393**	0.0142	0.6566	0.5421
(i= 4)	’(0.44)	’(-1.18)	’(-0.05)	’(-3.21)	’(4.30)	’(1.13)		

Note. Values shown in parentheses are t-values.

Because the consumption tax rate was increased in April 2014, the data that we used to estimate the parameters were from April 2015/2014 to December 2016/2015. Thus, each data type has 21 samples. The parameters estimated using OLS are shown in Table 2. All own price elasticities and income (total expenditure) elasticities have proper signs and high t-values.

[Second Stage]

In this stage, we estimate parameters in a specific month. We selected the months of October and November after considering the change in the consumption tax rate and seasonal variations (e.g., ceremonial usage and the rainy season, see also Table 1). Below, we explain the method that we use to estimate the parameters for October. The same method is also applied for November. Data used were the monthly changes from 2014 to 2015 and from 2015 to 2016, which are shown in Tables 3, 4, and 5. We assume that our parameters are the same in both periods to get steady results. Based on this assumption, the following Montgomery weights needed to be used. Given that the subscripts 4, 5, and 6 represent, respectively, 2014, 2015, and 2016, we have two demand functions from Eq. (44):

$$\Delta\log q_{54i} = \sum_j \varepsilon_{ij}\Delta\log p_{54j} + \eta_i\Delta\log y_{54} + \mu_i,$$

$$\Delta\log q_{65i} = \sum_j \varepsilon_{ij}\Delta\log p_{65j} + \eta_i\Delta\log y_{65} + \mu_i,$$

where

$$\Delta\log q_{54i} = \log q_{5i} - \log q_{4i}\text{, etc.}$$

Averaging two equations yields

$$A(\Delta\log q_i) = \sum_j \varepsilon_{ij}A(\Delta\log p_j) + \eta_i A(\Delta\log y) + \mu_i,$$

wherein A(.) represents the arithmetic mean, for example,

$$\mathrm{A}(\Delta\log q_i) = (\Delta\log q_{54i} + \Delta\log q_{65i})/2 = \Delta\log q_{64i}/2.$$

We used the three averages ($A\,(\Delta\log q_i), A\,(\Delta\log p_i)$, and $A\,(\Delta\log y)$) that are also shown in the tables above. Consequently, our budget constraint was

$$\Delta\log y_{64} = \sum \frac{L(m_{6i}, m_{4i})}{L(y_6, y_4)} \Delta\log m_{64i} = \sum \frac{L(m_{6i}, m_{4i})}{L(y_6, y_4)} (\Delta\log p_{64i} + \Delta\log q_{64i}),$$

or

$$\Delta\log y_{64}\;/\;2 = \sum \frac{L(m_{6i}, m_{4i})}{L(y_6, y_4)} (\Delta\log m_{64i}/2) = \sum \frac{L(m_{6i}, m_{4i})}{L(y_6, y_4)} (\Delta\log p_{64i}\;/\;2 + \Delta\log q_{64i}\;/\;2),$$

from which the Montgomery weights (2016/2014) shown in Table 5 were used. Below, the log-change values for demand, price, and total expenditure (e.g., $\Delta\log q_i$, $\Delta\log y$) are the averages shown in Tables 3 and 4, whereas the Montgomery weights are those (2016/14) shown in Table 5.

To estimate the parameters in Eq. (44), we made the most of a 4×6 matrix **A** given by

$$\mathbf{A} = \{a_{ij}\} = \begin{pmatrix} \varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{14}, \eta_1, \mu_1 \\ \cdots\cdots\cdots\cdots\cdots \\ \varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}, \varepsilon_{44}, \eta_4, \mu_4 \end{pmatrix}.$$

Our control variables are the three averages of the log-change values and the above-mentioned Montgomery weights, which are simply redefined as

$$\lambda_i \equiv \frac{L(m_i)}{L(y)} = \left(\frac{\Delta m_{tsi}}{\Delta\log m_{tsi}}\right) \Big/ \left(\frac{\Delta y_{ts}}{\Delta\log y_{ts}}\right),$$

in which the subscripts t and s may be deleted.

Table 3. Control Variables 1

		$\Delta\log q_i$			
		Beef	Pork	Chicken	Others
Oct.	2015/14	-0.03663	0.06442	0.04063	0.07227
	2016/15	0.07310	0.03196	0.04240	0.00951
	Average	**0.01824**	**0.04819**	**0.04152**	**0.04089**
Nov.	2015/14	-0.00122	0.06594	0.07037	0.00406
	2016/15	0.09503	0.03192	0.08082	-0.02774
	Average	**0.04691**	**0.04893**	**0.07560**	**-0.01184**

While we need to estimate 24 parameters for October, the number of control variables are 13: four log-change variables for each of demand, price, and weight of the budget constraint; and the log-change variable for total expenditure. Nevertheless, by applying the WRAS method to these initial values, we can produce desirable parameters. For details of the RAS (or biproportion) method, refer to Eurostat (2008, Subsection 14.3), de Mesnard (2011), and Note 7 (see also Bacharach, 1970, Chap. 3).

Table 4. Control Variables 2

		$\Delta\log p_i$				$\Delta\log y$
		Beef	Pork	Chicken	Others	
Oct.	2015/14	-0.03005	-0.02901	-0.02820	-0.01455	0.00217
	2016/15	-0.13590	-0.04122	-0.02902	-0.04879	-0.02092
	Average	**-0.08298**	**-0.03512**	**-0.02861**	**-0.03167**	**-0.00938**
Nov.	2015/14	-0.04552	-0.03459	-0.04400	0.02833	0.00803
	2016/15	-0.11927	-0.06092	-0.04493	-0.04600	-0.01582
	Average	**-0.08240**	**-0.04776**	**-0.04447**	**-0.00884**	**-0.00390**

Table 5. Control Variables 3

		$\lambda_i = L(m_i)/L(y)$				Total
		Beef	Pork	Chicken	Others	
Oct.	2015/14	0.2885	0.4245	0.2167	0.0702	0.9998
	2016/15	0.2729	0.4339	0.2216	0.0715	1.0000
	2016/14	**0.2827**	**0.4268**	**0.2206**	**0.0695**	**0.9996**
Nov.	2015/14	0.2849	0.4245	0.2189	0.0716	0.9999
	2016/15	0.2760	0.4268	0.2268	0.0704	0.9999
	2016/14	**0.2838**	**0.4218**	**0.2247**	**0.0696**	**0.9998**

Table 6. Rough Elements of Matrix A

	ε_{i1}	ε_{i2}	ε_{i3}	ε_{i4}	η_i	μ_i
Beef (i = 1)	-1.3	0.1	-0.1	-0.1	1	0
Pork (i = 2)	-0.1	-1.3	0.1	0.1	1	0
Chicken (i = 3)	-0.1	-0.1	-1.3	-0.1	1	0
Others (i = 4)	0.1	-0.1	-0.1	-1.3	1	0

First, we settled the rough elements of matrix **A**, as shown in Table 6. Each $\varepsilon_{ii} = -1.3, \varepsilon_{ij} = 0.1$ or -0.1 $(j \neq i), \eta_i = 1,$ and $\mu_i = 0$ (i and $j = 1, 2, 3, 4$). The signs of the price and income elasticities are the same as those in Table 2. All the initial values employed by our WRAS method must be positive. So, we modified the values presented in Table 6 by adding 5.0 and 0.2 to all of the own and cross price elasticities, respectively, and 1.1 to all of the residuals (Note 7). The additive values to the residuals are discussed below. Thus, the initial values (or elements) of matrix **A** (written as **A0** = $\{a0_{ij}\}$) are

$$\mathbf{A0} = \{a0_{ij}\} = \begin{pmatrix} 3.7, 0.3, 0.1, 0.1, 1, 1.1 \\ 0.1, 3.7, 0.3, 0.3, 1, 1.1 \\ 0.1, 0.1, 3.7, 0.1, 1, 1.1 \\ 0.3, 0.1, 0.1, 3.7, 1, 1.1 \end{pmatrix}. \text{(Note 8)} \quad (45)$$

We use our WRAS method to constrain the initial values sufficiently to satisfy the following four conditions: the M-Engel condition (32), the M-Cournot conditions (33), the M-Residual condition (34), and the M-Demand equation conditions (44).

To calculate the row and column totals for the matrices, we employ the following weights:

Row weights: $\Delta \log p_1, \Delta \log p_2, \Delta \log p_3, \Delta \log p_4, \Delta \log y, 1$;

Column weights: $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

The subscripts are 1 for beef, 2 for pork, 3 for chicken, and 4 for others (Note that the log-change values are the same as those mentioned above). As we modified the values in Table 6, our control totals, which are equal to the weighted row and column totals, have to be changed. Since we used the elements of matrix **A0** in Eq. (45) as the initial values, the right-hand sides of the M-Demand equation conditions in Eq. (44) are rewritten as follows. For simplicity, we write the values in Table 6 as ε_{ij}, η_i, and μ_i, and those of Matrix **A0** as $\varepsilon^*_{ij}, \eta^*_i$, and μ^*_i.

$$\sum_j \varepsilon^*_{ij} \Delta \log p_j + \eta^*_i \Delta \log y + \mu^*_i$$

$$= \sum_j \varepsilon_{ij} \Delta \log p_j + 0.2 \sum_j \Delta \log p_j + 4.8 \Delta \log p_i + \eta_i \Delta \log y + 1.1 \; (i = 1, 2, 3, 4).$$

Thus, one of conditions on the left-hand sides of the M-Demand equations, which is the *i*th weighted row total, should be changed to

$$\Delta \log q^\#_i = \Delta \log q_i + 0.2 \sum_j \Delta \log p_j + 4.8 \Delta \log p_i + 1.1 \; (i = 1, 2, 3, 4).$$

Our weighted column totals were also changed, and the left-hand sides of the M-Cournot conditions are

$$\sum_{i=1}^{4} \lambda_i \varepsilon^*_{ij} = \sum_{i=1}^{4} \lambda_i \varepsilon_{ij} + 0.2 \sum_{i=1}^{4} \lambda_i + 4.8 \lambda_j (j = 1, 2, 3, 4).$$

Thus, one of the elements on the right-hand sides of the M-Cournot conditions, which is the *j*th weighted column total, should be

$$\lambda^\#_j = -\lambda_j + 0.2 \sum_{i=1}^{4} \lambda_i + 4.8 \lambda_j = 3.8 \lambda_j + 0.2 \sum_{i=1}^{4} \lambda_i \; (j = 1, 2, 3, 4).$$

The right-hand side of the M-Engel condition, which is the fifth weighted column total, does not change since the income elasticities are not modified. We write this as $\lambda^\#_5 = 1$. The left-hand side of the M-Residual condition is

$$\sum_{i=1}^{4} \lambda_i \mu^*_i = \sum_{i=1}^{4} \lambda_i \mu_i + 1.1 \sum_{i=1}^{4} \lambda_i = 1.1 \sum_{i=1}^{4} \lambda_i.$$

Thus, the right-hand side of this condition, which is the sixth weighted column total, is

$$\lambda_6^{\#} = 1.1\sum_{i=1}^{4}\lambda_i.$$

Using the above-mentioned weights and the control totals, we apply the WRAS method to the initial values of matrix **A0** and obtain the elements of matrix **A1** and matrix **A2** below. The above-mentioned values such as $\Delta\log q_i^{\#}$ and $\lambda_j^{\#}$ are always used as the numerators to obtain the coefficients of the row and column constraints below.

1) First round

1.1) Row constraints

From the M-Demand equation conditions, we obtained the coefficient of the *i*th row constraint r_i:

$$r_i = \frac{\Delta\log q_i + 0.2\sum_{j=1}^{4}\Delta\log p_j + 4.8\Delta\log p_i + 1.1}{\sum_{j=1}^{4} a0_{ij}\Delta\log p_j + a0_{i5}\Delta\log y + a0_{i6}}\,(i = 1,2,3,4).$$

All coefficients must be positive. If not, we must reconsider the elements of matrix **A0**. Moreover, the additive values to the residuals are desirable as small as possible. (If the *j*th additive value is very large, r_j approaches 1 (Note 9). Thus, this constraint turns out to be inactive.) Using the coefficient $r_i > 0$, we determined matrix **A1** as follows:

$$\mathbf{A1} = \{a1_{ij}\} = \begin{pmatrix} r_1a0_{11}, r_1a0_{12}, r_1a0_{13}, r_1a0_{14}, r_1a0_{15}, r_1a0_{16} \\ r_2a0_{21}, r_2a0_{22}, r_2a0_{23}, r_2a0_{24}, r_2a0_{25}, r_2a0_{26} \\ r_3a0_{31}, r_3a0_{32}, r_3a0_{33}, r_3a0_{34}, r_3a0_{35}, r_3a0_{36} \\ r_4a0_{41}, r_4a0_{42}, r_4a0_{43}, r_4a0_{44}, r_4a0_{45}, r_4a0_{46} \end{pmatrix}.$$

1.2) Column constraints

Using the M-Cournot conditions and the elements of matrix **A1**, we obtained the coefficient of the *j*th column constraint s_j as follows:

$$s_j = \frac{3.8\lambda_j + 0.2\sum_{i=1}^{4}\lambda_i}{\sum_{i=1}^{4}\lambda_i a1_{ij}}\,(j = 1, 2, 3, 4).$$

Using the M-Engel condition and the elements of matrix **A1**, we obtained the coefficient of the fifth column constraint s_5 as follows:

$$s_5 = \frac{1}{\sum_{i=1}^{4}\lambda_i a1_{i5}}.$$

Similarly, we obtained the coefficient of the sixth column constraint s_6 from the M-Residual condition and the elements of matrix **A1** as follows:

$$s_6 = \frac{1.1\sum_{i=1}^{4}\lambda_i}{\sum_{i=1}^{4}\lambda_i a1_{i6}}.$$

If this constraint is redundant, we always set $s_6 = 1$. For more details, see the final results presented below.

All coefficients are positive because all $a1_{ij} > 0$. Using these coefficients and the elements of matrix **A1**, we determined the new matrix **A2** as follows:

$$\mathbf{A2} = \{a2_{ij}\} = \begin{pmatrix} a1_{11}s_1, a1_{12}s_2, a1_{13}s_3, a1_{14}s_4, a1_{15}s_5, a1_{16}s_6 \\ a1_{21}s_1, a1_{22}s_2, a1_{23}s_3, a1_{24}s_4, a1_{25}s_5, a1_{26}s_6 \\ a1_{31}s_1, a1_{32}s_2, a1_{33}s_3, a1_{34}s_4, a1_{35}s_5, a1_{36}s_6 \\ a1_{41}s_1, a1_{42}s_2, a1_{43}s_3, a1_{44}s_4, a1_{45}s_5, a1_{46}s_6 \end{pmatrix}.$$

2) Second round and matrix **AX**

Using the elements of matrix **A2** and the coefficients of the row and column constraints calculated using the procedures outlined above, we determined matrices **A3** and **A4.** Repeating these computations, we obtained matrices **A5**, **A6**,..., **AX**, wherein **X** = 2*x* (*x* = 4, 5, …).We repeated these computations until all the elements of matrix **AX** were nearly equal to those of matrices $\mathbf{AX}_{-2}$ and $\mathbf{AX}_{-1}$ ($\mathbf{X}_{-2}$ = **X** - 2, $\mathbf{X}_{-1}$ = **X** - 1). This scenario occurred at about **X** = 100. To get stable results, we employed the matrix with **X** = 800 (i.e., **A800**). Below, we call these values the convergent elements.

Table 7. Convergent Elements of Matrix AX (October with the sixth column constraint)

	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
Beef (*i* = 1)	4.1469	0.3040	0.1009	0.1004	0.9586	1.0540
Pork (*i* = 2)	0.1194	3.9945	0.3224	0.3208	1.0213	1.1229
Chicken (*i* = 3)	0.1177	0.1064	3.9193	0.1054	1.0066	1.1068
Others (*i* = 4)	0.3588	0.1081	0.1077	3.9635	1.0230	1.1248

Table 8. Convergent Elements of Matrix AX (October without the sixth column constraint)

	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
Beef (*i* = 1)	4.1468	0.3040	0.1009	0.1004	0.9585	1.0541
Pork (*i* = 2)	0.1194	3.9945	0.3224	0.3208	1.0213	1.1231
Chicken (*i* = 3)	0.1177	0.1064	3.9193	0.1054	1.0066	1.1069
Others (*i* = 4)	0.3588	0.1081	0.1077	3.9635	1.0230	1.1250

When we repeated these computations, the sixth column constraint was not effective. To be specific, we computed two cases for October: one with the sixth column constraint and the other without this constraint. The results are shown in Tables 7 and 8, respectively. It can be seen that the convergent elements of the matrices are very similar. Similar results were gotten using data for November.

Table 9. Estimated Values (October without the sixth column constraint)

	ε_{i1}	ε_{i2}	ε_{i3}	ε_{i4}	η_i	μ_i
Beef (i = 1)	-0.8532	0.1040	-0.0991	-0.0996	0.9585	-0.0459
Pork (i = 2)	-0.0806	-1.0055	0.1224	0.1208	1.0213	0.0231
Chicken (i = 3)	-0.0823	-0.0936	-1.0807	-0.0946	1.0066	0.0069
Others (i = 4)	0.1588	-0.0919	-0.0923	-1.0365	1.0230	0.0250

Table 10. Estimated Values (November without the sixth column constraint)

	ε_{i1}	ε_{i2}	ε_{i3}	ε_{i4}	η_i	μ_i
Beef (i = 1)	-0.8409	0.1168	-0.0972	-0.0942	0.9837	-0.0181
Pork (i = 2)	-0.0853	-1.0148	0.1145	0.1237	1.0034	0.0035
Chicken (i = 3)	-0.0836	-0.0907	-1.0621	-0.0904	1.0187	0.0203
Others (i = 4)	0.1382	-0.0941	-0.0969	-1.0738	0.9867	-0.0149

We adopted the convergent elements without the sixth column constraint as the final results. Subtracting each additive value from the corresponding element in Table 8, we got the elasticities and the residuals, which are shown in Table 9. These values satisfy the M-Demand equation conditions (44), the M-Engel condition (32), and the M-Cournot conditions (33). They also satisfy the M-Residual condition (34) as follows:

$$\sum \lambda_i \mu_i = 0.00014,$$

the value of which was negligible in our computations. The estimated own price elasticities and income elasticities appear to be reasonable and the estimated cross price elasticities have the same signs as those in Tables 2 and 6.

Hence, we can say that our estimated parameters are consistent with real data, which has the following two implications (also see the next section). Substituting these parameters and real data such as p_i and y on the right-hand side of Eq. (44), we obtain the theoretical demand for each commodity that equals the real (or actual) value. Substituting these parameters and the Montgomery weights on the left-hand sides of Eqs. (32), (33), and (34), we get the corresponding values that equal the real values on the right-hand sides, respectively (Besides, we get the budget constraint (31) from Eqs. (44), (32), (33), and (34); that is, these parameters satisfy the budget constraint).

Similarly, we adopted the convergent elements without the sixth column constraint as the final results for November and got their elasticities and residuals. The results are shown in Table 10, and all values also satisfy the four above-mentioned conditions. Thus, the estimated parameters for November are also consistent with real data. All the estimated elasticities have the same signs as those shown in Table 9.

Additionally, each estimated elasticity in October is similar to that in November, but the residuals differ (e.g., $\varepsilon\_11 = -0.8532$ in Oct. vs. $\varepsilon\_11 = -0.8409$ in" $Nov.$", $\mu\_1 = -0.0459$ in" $Oct. vs.$" $\mu\_1 = -0.0181$ "$in\ Nov.$)." To gain more accurate results, the procedures that we use to obtain the initial values in matrix **A** are crucial. (Strictly speaking, we cannot assess whether these estimated elasticities are reasonable and have the proper signs because we have never tried to estimate these parameters). It may be helpful to compare our results with those obtained using the differential approach (e.g., Fousekis & Revell, 2000; Okrent & Alston, 2012 (Appendix-Table A.4)) and other approaches (e.g., Hayes, Wahl, & Williams, 1990).

6. Concluding Remarks

Some differential demand systems or their variants, such as the Rotterdam model, are well-known. However, we cannot derive a differential demand system that is consistent with real data, as the differentials such as dx and $d\log x$ (x is any economic datum) are unable to be observed or measured. Thus, we must develop an alternative approach, that is, the difference demand system.

Various correspondences between the differential and difference approaches (or calculi) are presented by Tsuchida (2018), and this study extends and evolves these correspondences. Specifically, we have shown a difference quotient corresponding to a differential quotient, which is generally called a derivative, and a partial difference quotient corresponding to a partial differential quotient, which is generally called a partial derivative. From these, we have derived a continuous (i.e., differential) log-change demand function with point elasticities as parameters and a discrete (i.e., difference) log-change demand function with logarithmic mean elasticities as parameters. Based on these results, we have defined continuous and discrete demand systems that should satisfy each budget constraint. We can also apply these demand systems to any group of commodities (e.g., a meat demand system).

Our discrete meat demand system was applied to monthly demand for fresh meat (beef, pork, chicken, and other meats) in Japan, and its parameters (elasticities and residuals) were estimated using the weighted RAS (WRAS) method, which is handy and practical. Whereas 24 parameters must be estimated in a given month, 13 control variables are used in our method. Nevertheless, our WRAS method can derive a discrete meat demand system in which the estimated parameters are consistent with real data. This implies that each theoretical value of the conditional demand functions calculated using estimated parameters and independent variables coincides with its real value (observed demand), and each set of these parameters satisfies the Engel, Cournot, and Residual conditions.

As the difference approach to the demand system and its estimating method have scarcely been studied, we offer several remarks (see also the difference version of the AIDS in Appendix A).

1) We begin by reconsidering the level of consistency with real data, on which we place particular emphasis. For an alternative explanation, we use matrix algebra. Our estimated parameters in October shown in Table 9 are given as a 4×6 matrix $\mathbf{B} = \{b_{ij}\}$. For example, $b_{12} = \varepsilon_{12}$ and $b_{45} = \eta_4$. Our real data are given by vectors. For example, any vector $\mathbf{x}$ and its transpose $\mathbf{x}^t$ are given by

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \mathbf{x^t} = (2, 1).$$

Our four vectors in October are

$$\mathbf{u^t} = \left(\Delta \log p_1, \Delta \log p_2, \Delta \log p_3, \Delta \log p_4, \Delta \log y, 1\right),$$

$$\mathbf{q^t} = \left(\Delta \log q_1, \Delta \log q_2, \Delta \log q_3, \Delta \log q_4\right),$$

$$\boldsymbol{\lambda}^\mathbf{t} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4), \mathbf{v^t} = (-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4, 1, 0),$$

wherein the subscripts 1, 2, 3, and 4 represent beef, pork, chicken, and others (i.e., other meats), respectively. From these, we obtain two equations:

$$\mathbf{q} = \mathbf{Bu}, \quad (46)$$

$$\mathbf{v^t} = \boldsymbol{\lambda}^\mathbf{t}\mathbf{B}. \quad (47)$$

Eq. (46) is the same as Eq. (44), and Eq. (47) turns out to be a combination of Eqs. (32), (33), and (34). Thus, Eqs. (46) and (47) indicate that matrix B (the estimated parameters) is consistent with the four vectors (real data). Similar results are obtained using data for November.

2) While the normal and continuous demand systems are based on economic theory (see, e.g., Barten, 1977; Piggott & Marsh, 2011), the discrete demand system is not. Our discrete demand system can only be derived from a specific demand system based on economic theory. Thus, we have to investigate how to combine economic theory with the discrete demand system. First, we define a utility function and derive a demand system based on the relevant theory. Next, we derive the discrete demand system from this system and estimate its parameters using the WRAS method.

3) Additionally, we have to investigate how to identify a utility function whose parameters have coherent properties to those produced using our discrete demand system. As examples to check this coherence, we explain the difference versions of the linear expenditure system and almost ideal demand system in Appendix A.

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Appendix A: Differential and Difference Versions of the LES and AIDS

The linear expenditure system (LES) (see, e.g., Stone, 1954; Phlips, 1974; Theil, 1975/76) is given by

$$p_i q_i = p_i b_i + a_i y - a_i \sum_j p_j b_j \quad (i \text{ and } j = 1, 2, \ldots, n), \tag{A1}$$

wherein $1 > a_i > 0, \sum a_i = 1, q_i > b_i > 0,$ and $y = \sum p_i q_i$; and a_i and b_i are constants. The properties of parameters a_i and b_i follow those of the Klein-Rubin (or Stone-Geary) utility function. The LES embeds the budget constraint within itself, and thus the differential and difference versions of it satisfy their budget constraints (see Subsection 4.3). Total differentiation leads to the following:

$$q_i dp_i + p_i dq_i = b_i dp_i + a_i dy - a_i \sum_j b_j dp_j,$$

$$p_i q_i d\log p_i + p_i q_i d\log q_i = b_i p_i d\log p_i + a_i y d\log y - a_i \sum_j b_j p_j d\log p_j,$$

$$\therefore d\log q_i = \left(\frac{b_i p_i - p_i q_i}{p_i q_i}\right) d\log p_i - \frac{a_i}{p_i q_i}\sum_j b_j p_j d\log p_j + \frac{a_i y}{p_i q_i} d\log y. \quad \text{(A2)}$$

This is the differential version of the LES. The point elasticities in Eq. (A2) are

$$e_{ii} = \frac{b_i p_i (1 - a_i)}{p_i q_i} - 1, e_{ij} = -\frac{a_i b_j p_j}{p_i q_i} (j \neq i), h_i = \frac{a_i y}{p_i q_i} = \frac{a_i}{w_i}.$$

These elasticities satisfy the Homogeneity, Engel, and Cournot conditions explained in Subsection 4.2. These proofs are easy, and thus are omitted. From the initial assumptions, $-1 < e_{ii} < 0$ and $e_{ij} < 0$ ($j \neq i$). For these point elasticities and the theoretical properties for the LES, refer to Phlips (1974, Subsection 4.3.3), Theil (1975/76, Chaps 1,3, etc.), and Deaton and Muellbauer (1980b, Chap. 3).

These elasticities are point-dependent. Assuming that (A1) was estimated using yearly data from 1996 to 2015, we have 20 sets of elasticities from the above relationships. Note that p_i, q_i, and y are yearly data; and a_i and b_i are constants.

The LES at time *t* is

$$p_{ti} q_{ti} = p_{ti} b_i + a_i y_t - a_i \left(\sum_j p_{tj} b_j\right) \quad (i \text{ and } j = 1, 2, \dots, n),$$

wherein $q_{ti} > b_i > 0$ and $y_t = \Sigma p_{ti} q_{ti}$. While the differential approach produces the unique outcome shown above, the difference approach may produce multiple outcomes, as discussed in Tsuchida (2018). We have two ideal log-change indices that are derived using Transformation-M ($m_{ti} = p_{ti} q_{ti}$) and Transformation-W($w_{ti} = p_{ti} q_{ti} / y_t$). We can apply these transformations to the LES, and thus two difference versions are obtained. First, we utilize Transformation-M.

Substituting m_{ti} and $k_{ti} = p_{ti} b_i$ into the above LES, we have:

$$m_{ti} = k_{ti} + a_i \left(y_t - \sum_j k_{tj}\right), \Delta m_i = \Delta k_i + a_i \left(\Delta y - \sum_j \Delta k_j\right),$$

$$\Delta \log m_i = \Delta \log p_i + \Delta \log q_i, \Delta m_i = L(m_i) \Delta \log p_i + L(m_i) \Delta \log q_i,$$

$$\Delta k_i = b_i \Delta p_i = b_i L(p_i) \Delta \log p_i, \Delta y = L(y) \Delta \log y.$$

$$\therefore \Delta \log q_i = \left(\frac{b_i L(p_i) - L(m_i)}{L(m_i)}\right) \Delta \log p_i - \frac{a_i}{L(m_i)} \sum_j b_j L(p_j) \Delta \log p_j + \frac{a_i L(y)}{L(m_i)} \Delta \log y.$$

Note that $\Delta \log q_i = \log q_{ti} - \log q_{si}, L(p_i) = L(p_{ti}, p_{si})$, etc., and $s = t - 1$. This is one of the difference versions of the LES. The log-mean elasticities, which were named the M-elasticities, are

$$\varepsilon_{ii} = \frac{b_i L(p_i)(1 - a_i)}{L(m_i)} - 1, \varepsilon_{ij} = -\frac{a_i b_j L(p_j)}{L(m_i)} (j \neq i), \eta_i = \frac{a_i L(y)}{L(m_i)}.$$

These elasticities are twop-dependent and satisfy Eqs. (32) and (33) (see Subsection 4.3). These proofs are very easy, and thus are omitted. When Eq. (A1) is estimated using yearly data as the above, we have 19 sets of elasticities.

If we assume that the discrete meat demand system (44) is based on the LES, we have the following discrete meat demand system:

$$\Delta \log q_i = \sum_j \varepsilon_{ij} \Delta \log p_j + \eta_i \Delta \log y + \mu_i.$$

The price and income elasticities are the same as those outlined above and the residuals μ_i satisfy the M-Residual condition (34). These 19 sets of elasticities and residuals can also be estimated using the WRAS method, and should coincide with the above M-elasticities calculated using a_i, b_i, and various log-means. Hence, we can determine whether our assumption is plausible. Applying Transformation-W in the below, we can similarly do.

Next, we discuss another version using Transformation-W. From the LES, we have:

$$\frac{m_{ti}}{y_t} = \frac{k_{ti}}{y_t} + a_i \left(1 - \sum_j \frac{k_{tj}}{y_t}\right), w_{ti} = v_{ti} + a_i \left(1 - \sum_j v_{tj}\right),$$

wherein $v_{ti} = k_{ti}/y_t = b_i p_{ti}/y_t$. Hence,

$$\Delta w_i = \Delta v_i - a_i \sum_j \Delta v_j, \quad \Delta w_i = L(w_i) \Delta \log w_i = L(v_i) \frac{\Delta v_i}{L(v_i)} - a_i \sum_j L(v_j) \frac{\Delta v_j}{L(v_j)},$$

$$\Delta \log w_i = \Delta \log p_i + \Delta \log q_i - \Delta \log y, \Delta \log v_i = \Delta \log k_i - \Delta \log y = \Delta \log p_i - \Delta \log y.$$

From this, we obtain

$$\Delta \log q_i = \left(\frac{L(v_i)}{L(w_i)} - 1\right) \Delta \log p_i - \frac{a_i}{L(w_i)} \sum_j L(v_j) \Delta \log p_j$$
$$+ \left(1 - \frac{L(v_i)}{L(w_i)} + \frac{a_i}{L(w_i)} \sum_j L(v_j)\right) \Delta \log y.$$

This is another difference version of the LES, whose own and cross price elasticities and income elasticities, which are called W-elasticities, are

$$\varepsilon_{ii}^w = \frac{L(v_i)}{L(w_i)}(1 - a_i) - 1, \varepsilon_{ij}^w = \frac{-a_i L(v_j)}{L(w_i)} (j \neq i), \eta_i^w = 1 - \frac{L(v_i)}{L(w_i)} + \frac{a_i}{L(w_i)} \sum_j L(v_j).$$

These W-elasticities differ from the M-elasticities.

Based on the other budget constraint using Transformation-W ($1 = \sum w_{ti}$), we obtain

$$0 = \sum_i \Delta w_i = \sum_i L(w_i) \Delta \log w_i = \sum_i L(w_i) \left(\Delta \log p_i + \Delta \log q_i - \log y\right),$$

$$\therefore \ \Delta \log y = \sum_i \frac{L(w_i)}{\sum_k L(w_k)} \ \Delta \log p_i + \sum_i \frac{L(w_i)}{\sum_k L(w_k)} \Delta \log q_i. \qquad \text{(A3)}$$

From Eq. (A3), we find the Vartia-Sato index. The first and second terms of the right-hand side are, respectively, the ideal log-change price and quantity indices (Note 10). This budget constraint (A3) produces the two conditions related to W-elasticities as follows:

$$\sum_i \big(L(w_i)/\sum_k L(w_k)\big)\eta_i^w = 1\big(i \text{ and } k = 1, 2, \dots, n\big), \text{ (Engel condition);} \quad \text{(A4)}$$

$$\sum_i \big(L(w_i)/\sum_k L(w_k)\big)\varepsilon_{ij}^w = -\Big(L\big(w_j\big)/\sum_k L(w_k)\Big)\big(i, j, \text{and } k = 1, 2, \dots, n\big), \text{ (Cournot conditions).} \quad \text{(A5)}$$

The W-elasticities also satisfy Eqs. (A4) and (A5). These proofs are also easy. Note that these aggregation weights add up to 1and differ from those of the M-elasticities.

If the ratio of two positive variables, x_1 and x_0, is close to unity (i.e., $x_1/x_0 \approx 1$), we have a good relationship (Tsuchida, 2014, 2018):

$$L(x) \approx G(x), L(x) \approx A(x), A(x) \approx G(x),$$

wherein $G(x) = (x_0x_1)^{0.5}$ is the geometric mean and $A(x)$ is, as mentioned above, the arithmetic mean. If all variables below satisfy these assumptions, we obtain the following approximations:

$$L(m_i)/L(y) \approx G(m_i)/G(y) = G(w_i) \approx A(w_i) \approx L(w_i),$$

$$\therefore \frac{L(m_i)}{L(y)} \approx A(w_i) \approx \frac{A(w_i)}{\sum_k A(w_k)} \approx \frac{L(w_i)}{\sum_k L(w_k)} \quad (i \text{ and } k = 1,2, \dots, n)$$

Thus, the weights derived using Transformation-M approach those derived using Transformation-W.

Finally, we briefly discuss the almost ideal demand system (AIDS; Deaton & Muellbauer, 1980a, 1980b; Barten, 1993). The AIDS is

$$w_i = \alpha_i + \sum_j \gamma_{ij} \log p_j + \beta_i \big(\log y - \log P\big)\big(i \text{ and } j = 1, 2, \dots, n\big), \quad \text{(A6)}$$

wherein P is given by

$$\log P = \alpha_0 + \sum \alpha_k \log p_k + \frac{1}{2}\sum_j \sum_k \gamma_{kj} \log p_k \log p_j \ \big(k \text{ and } j = 1, 2, \dots, n\big). \quad \text{(A7)}$$

An explanation of the constraints regarding the parameters in (A6) and (A7) is omitted. Differentiating (A6) leads to

$$dw_i = \sum_j \gamma_{ij} d\log p_j + \beta_i \big(d\log y - d\log P\big)\big(i \text{ and } j = 1, 2, \dots, n\big). \quad \text{(A8)}$$

Substituting (A9) and (A10) into (A8), we get the differential demand system.

$$dw_i = w_i d\log w_i = w_i(d\log p_i + d\log q_i - d\log y), \quad \text{(A9)}$$

$$d\log P \approx \sum w_i d\log p_i \,. \quad \text{(A10)}$$

Note that the right-hand side of (A10) is the price term of the differential budget constraint (30).

The difference version of (A6) is

$$\Delta w_i = \sum_j \gamma_{ij} \Delta\log p_j + \beta_i \big(\Delta\log y - \Delta\log P\big)\big(i \text{ and } j = 1, 2, \dots, n\big). \quad \text{(A11)}$$

Similarly, substituting (A12) and (A13) into (A11), we obtain the difference demand system.

$$\Delta w_i = L(w_i)\Delta\log w_i = L(w_i)(\Delta\log p_i + \Delta\log q_i - \Delta\log y), \quad \text{(A12)}$$

$$\Delta \log P \approx \sum \frac{L(w_i)}{\sum_k L(w_k)} \Delta \log p_i \,. \quad \text{(A13)}$$

The above (A12) corresponds to (A9) and (A13) corresponds to (A10), though the right-hand side of (A13) is the price term of the difference budget constraint (A3). This system is produced using Transformation-W. (If we use Transformation-M, we have to multiply both sides of (A6) by y, whereby we have numerous non-linear terms.)

Deaton and Muellbauer (1980a) use the following approximation (A14) instead of (A13):

$$\Delta \log P \approx \sum w_i \,\Delta \log p_i. \quad \text{(A14)}$$

This may be regarded as an approximation of (A10), but is neither the price term of the difference budget constraint (A3) nor that of Theil's approximation (B2) in Appendix B. They also show another difference version (Deaton & Muellbauer, 1980a, Eq. (21)).

Given that Eq. (A6) leads to our meat demand system, its discrete meat demand system is given by

$$\Delta \log q_i = \sum_j \varepsilon_{ij}^w \,\Delta\log p_j + \eta_i^w \Delta \log y + \mu_i^w, (i \text{ and } j = 1, 2, 3, 4), \quad \text{(A15)}$$

wherein the parameters are

$$\varepsilon_{ij}^w = \frac{\gamma_{ij}}{L(w_i)} - \delta_{ij}, \eta_i^w = \frac{\beta_i}{L(w_i)} + 1, \mu_i^w = \mu_i - \frac{\beta_i}{L(w_i)} \Delta \log P.$$

Here, δ_{ij} is the Kronecker delta, which takes 1 if $i = j$, and 0 otherwise; and μ_i is the residual and includes an error that is accompanied by the approximation (A13). From the budget constraint (A3), we have the Engel condition (A4) and Cournot conditions (A5); and the Residual condition that is given by

$$\frac{\sum_i L(w_i)\mu_i^w}{\sum_k L(w_k)} = \frac{\sum_i L(w_i)\mu_i}{\sum_k L(w_k)} = 0.$$

Hence, we can estimate these parameters in Eq. (A15) using our WRAS method. Based on those results and some log-means, we get the parameters γ_{ij} and β_i in (A6). Thus, we can check whether some estimated parameters have the properties coherent to those of its utility function.

Appendix B: Theil's Approximation to the Differential Budget Constraint

We showed the approximation formed from Eqs. (24) and (26), which is rewritten as the following (B1):

$$\Delta \log q_i \approx \sum_j e_{ij} \Delta \log p_j + h_i \Delta \log y \quad (i \text{ and } j = 1, 2, \ldots, n). \quad \text{(B1)}$$

Although e_{ij} and h_i are the point elasticities, we should utilize the difference budget constraints such as Eq. (31) or (A3) to derive the Engel and Cournot conditions for finite-change variables. Nevertheless, Theil's approximation to the differential budget constraint (30) is commonly utilized as expressed in Subsection 4.2. This approximation is rewritten as follows:

$$\Delta \log y \approx \sum_i (A(w_i)\,\Delta \log p_i + A(w_i)\Delta \log q_i), \quad \text{(B2)}$$

wherein $A(w_i) = (w_{ti} + w_{t-1i})/2.$ To employ the approximation (B2), the differences of all of the variables (w_i, y, p_i, and q_i) should be very small. The budget constraint (B2) produces two budget conditions as follows:

$$\sum_i A(w_i)\eta_i \approx 1(i = 1, 2, \dots, n), \text{ (Engel condition);} \quad \text{(B3)}$$

$$\sum_i A(w_i)\varepsilon_{ij} \approx -A(w_j)(i \text{ and } j = 1, 2, \dots, n), \text{(Cournot conditions).} \quad \text{(B4)}$$

wherein the two elasticities are M-elasticities (they may also be W-elasticities). Note that $A(w_i)$ and the elasticities are dependent on two points. The two elasticities in (B1) will approximately satisfy the two conditions (B3) and (B4), if we assume that the elasticities e_{ij} and h_i in (B1) are close to the log-mean elasticities ε_{ij} and $\eta_i$in (B4) and (B3). Given that a double-log function produces this continuous demand system, the assumptions are fulfilled. Recall that all double-log functions produce point elasticities that are always equal to the log-mean elasticities discussed in Subsection 2.2.

Appendix C: Differential and Difference Approaches for a Composite Function

There are some functions for which we cannot always derive the correspondence between the differential and difference approaches (see Tsuchida, 2018). An example is presented below.

$$Y = X \log Z \text{ and } Y_t = X_t \log Z_t.$$

The differential approach quickly derives:

$$dY = \log Z\, dX + Xd \log Z = X \log Z\, d \log X + Xd \log Z = Yd \log X + (Y/\log Z)d \log Z. \quad \text{(C1)}$$

Note that all variables are positive and not 1 when we take their logarithms.

The difference approach needs a further assumption, that is, $\log Z_t > 0$ $(\text{i.e., } Z_t > 1)$ to maintain correspondence. Then, the difference approach yields:

$$\log Y_t = \log X_t + \log(\log Z_t), \Delta\log Y = \Delta \log X + \Delta \log \log Z = \Delta \log X + \Delta \log Z/(L(\log Z)),$$

wherein

$$\Delta \log \log Z = \log(\log Z_1) - \log(\log Z_0), L(\log Z) = \Delta \log Z/(\Delta \log \log Z).$$

$$\therefore \Delta Y = L(Y)\Delta\log X + (L(Y)/L(\log Z))\Delta \log Z. \quad \text{(C2)}$$

The correspondences between (C1) and (C2) can be found only when $Z > 1$ and $Z_t > 1$. Hence, we cannot always derive the difference version for a demand function that has a composite explanatory variable such as that shown above.

Appendix D: Proofs of Some Properties of a Logarithmic Mean

We restate Montgomery weights inequality as follows:

$$\sum_i \frac{L(m_{1i}, m_{0i})}{L(y_1, y_0)} = \sum_i \frac{L(m_{1i}, m_{0i})}{L(\sum_j m_{1j}, \sum_j m_{0j})} \leq 1, \ (i \text{ and } j = 1, 2, \ldots, n). \quad \text{(D1)}$$

If $m_{1i} = m_{0i}$ for all i, the inequality turns out to be the identity. The log-mean has the following property:

$$L(x_1, x_0) = \frac{x_1^{1/2} + x_0^{1/2}}{2} \frac{x_1^{1/2} - x_0^{1/2}}{(1/2)\log(x_1/x_0)} = A(x_1^{1/2}, x_0^{1/2})L(x_1^{1/2}, x_0^{1/2})$$
$$= A(x_1^{1/2}, x_0^{1/2})A(x_1^{1/4}, x_0^{1/4}) \ldots A(x_1^{1/z}, x_0^{1/z}), \quad \text{(D2)}$$

wherein $z = 2^n (n = 3, 4, 5, \ldots)$.The last equation is obtained by expanding the first equality of (D2) repeatedly and the same as the Corollary 2 in Carlson (1972). Moreover, we have two relationships:

$$L(m_{1i}, m_{0i}) = m_{0i}L(m_{1i}/m_{0i}, 1) = m_{0i}L(k_i, 1),$$
$$L(y_1, y_0) = y_0L(y_1/y_0, 1) = y_0L\left(\frac{\sum_i m_{0i}(m_{1i}/m_{0i})}{y_0}, 1\right) = y_0L\left(\sum_i w_{0i}k_i, 1\right),$$

wherein $0 < k_i = m_{1i}/m_{0i}$ and $0 < w_{0i} = m_{0i}/y_0$. Using these two relationships and Eq. (D2), we produce

$$\sum_i m_{0i}L(k_i, 1) = \sum_1 m_{0i}A(k_i^{1/2}, 1)A(k_i^{1/4}, 1) \ldots = \sum m_{0i}\left(\frac{k_i^{1/2}+1}{2}\right)\left(\frac{k_i^{1/4}+1}{2}\right) \ldots, \text{(Note 11)}$$
$$y_0L\left(\sum_i w_{0i}k_i, 1\right) = y_0\left(\frac{(\sum w_{0i}k_i)^{1/2}+1}{2}\right)\left(\frac{(\sum w_{0i}k_i)^{1/4}+1}{2}\right) \ldots.$$

Substituting these into the left-hand side of the inequality (D1), we obtain

$$\sum_i \frac{L(m_{1i}, m_{0i})}{L(y_1, y_0)} = \sum_i \frac{m_{0i}((k_i^{1/2}+1)/2)((k_i^{1/4}+1)/2) \ldots}{y_0(((\sum w_{0i}k_i)^{1/2}+1)/2)(((\sum w_{0i}k_i)^{1/4}+1)/2) \ldots}$$
$$= \sum_i \frac{w_{0i}(k_i^{3/4}+k_i^{2/4}+k_i^{1/4}+1) \ldots}{(\sum w_{0i}k_i)^{3/4}+(\sum w_{0i}k_i)^{2/4}+(\sum w_{0i}k_i)^{1/4}+1 \ldots} \text{ (Note 12)}$$
$$= \frac{\sum w_{0i}k_i^{3/4}+\sum w_{0i}k_i^{2/4}+\sum w_{0i}k_i^{1/4}+1}{(\sum w_{0i}k_i)^{3/4}+(\sum w_{0i}k_i)^{2/4}+(\sum w_{0i}k_i)^{1/4}+1} \ldots.$$

Hence, we can apply Hölder's inequality (Berck & Sydsæter, 1991) given by

$$\sum_{i=1}^{n} a_ib_i \leq \left(\sum_{I=1}^{n} a_i^p\right)^{1/p}\left(\sum_{i=1}^{n} b_i^q\right)^{1/q},$$

wherein a_i and b_i are positive; and $p > 1$, $q > 1$, and $1/p + 1/q = 1$. Using the inequality, we produce the following inequalities:

1) If $a_i = w_{0i}^{1/4}, b_i = w_{0i}^{3/4}k_i^{3/4}, p = 4,$ and $q = 4/3$, then $\sum w_{0i}k_i^{3/4} \leq \left(\sum w_{0i}k_i\right)^{3/4}$;

2) If $a_i = w_{0i}^{2/4}, b_i = w_{0i}^{2/4}k_i^{2/4},$ and $p = q = 4/2$, then $\sum w_{0i}k_i^{2/4} \leq \left(\sum w_{0i}k_i\right)^{2/4}$;

3) If $a_i = {w_{0i}}^{3/4}, b_i = {w_{0i}}^{1/4}{k_i}^{1/4}, p = 4/3$, and $q = 4$, then $\sum w_{0i}{k_i}^{1/4} \leq \left(\sum w_{0i}k_i\right)^{1/4}$.

Using these inequalities and comparing each term in the numerator of the above equations with the corresponding term in the denominator, we find the inequality (D1). Whereas we only use the values composed of the first and second terms of the last expansion of Eq. (D2), we can apply Hölder's inequality to those terms including the third and higher terms, for which comparisons we use the bellow.

When we compare $\sum w_{0i}k_i^{1/q}$ with $(\sum w_{0i}k_i)^{1/q}$, we employ the following approach. Letting $a_i = w_{0i}^{1/p}$ and $b_i = w_{0i}^{1/q}k_i^{1/q}$, we have

$$\sum a_i b_i = \sum w_{0i}k_i^{1/q} \leq (\sum a_i^p)^{1/p}(\sum b_i^q)^{1/q} = (\sum w_{0i}k_i)^{1/q}.$$

These results are helpful for any comparison. For example, we obtained the above a_i and b_i using the following p and q:

In 1) above: $1/q = 3/4, 1/p = 1 - 3/4, \therefore p = 4$ and $q = 4/3$.

In 2) above: $1/q = 2/4, 1/p = 1 - 2/4, \therefore p = q = 4/2$.

In 3) above: $1/q = 1/4, 1/p = 1 - 1/4, \therefore p = 4/3$ and $q = 4$.

Appendix E: Data Used and Some Compilations

We used data from the *annual report on the family income and expenditure survey* (2014-2016) and the *annual report on the consumer price index* (CPI, 2014-2016) (both published by the Statistics Bureau, Ministry of Internal Affairs and Communications, Japan). Monthly data on expenditure, quantity, and average price per household for each commodity were extracted from Table 3 (2014) and Table 10 (2015 and 2016) in the former reports. Our *others* category for fresh (or raw) meat was the sum of mixed ground meat and other raw meat. As these tables show the number of persons per household, we calculated per capita expenditure and quantity. As 2016 was a leap year, the expenditure and quantity figures reported for February 2016 were multiplied by 28/29.

Each reported price was deflated by the CPI, in which the bench-mark years were changed from 2014 to 2016. We have two non-connected datasets as the CPI: 2014-2015 (bench-mark year is 2010, shortly 2010-base) and 2015-2016 (bench-mark year is 2015, shortly 2015-base). The fresh meat items reported in the 2010-base are beef A, beef B, pork A, pork B, chicken, and liver. (The weight of liver is very small, so this is discarded.) Those reported in the 2015-base are the items other than liver. The weights of the items in 2010-base series are slightly different from those in 2015-base series. We used the weighted averages of each of the two beef and pork items as our price indices for beef and pork, respectively, and the price index for raw meats as our index for others.

All of our indices were 2015-base after disregarding the above-mentioned differences. The indices in 2016 were those presented in Table 4-1 in the CPI (2016), whereas those in 2014 were calculated using the reciprocal of the 2015/2014 indices (percentage changes over the previous year) presented in Table 7-1 in the CPI (2015).

Notes

Note 1. If the functions of three or more variables (X, Z, W …) are considered, we have to rewrite the condition as $dZ = dW = \ldots = 0$.

Note 2. If the functions of three or more variables (X_t, Z_t, W_t …) are considered, we have to rewrite the condition as $\Delta Z = \Delta W = \ldots = 0$.

Note 3. The almost ideal demand system (AIDS) has a nonlinear price term, which is usually approximated to a linear price term to estimate each demand function. Its differential system is derived from this *linear* AIDS (Deaton & Muellbauer, 1980a, 1980b; Barten, 1993). Our difference system can be derived using this linear term (see Appendix A).

Note 4. Frisch (1959) called these the Engel aggregation and Cournot aggregations.

Note 5. The Montgomery weights are not always able to derive Eq. (43). However, the primitive difference demand system below produces these weights that add up to 1 because of $L(m_{ti}, m_{si})/L(y_t . y_s) = (m_{ti} - m_{si})/(y_t - y_s)$ ($\because \Delta \log y_{ts} = \Delta \log m_{tsi}$).

Note 6. For these reasons, we omit the superscripts t and s from the elasticities, other than in this subsection.

Note 7. Our method for handling negative values is very easy and can produce negative elasticities in the final results. Compare ours with the GRAS method used by Junius & Oosterhaven (2003) and the IGRAS method used by Huang, Kobayashi, & Tanji (2008). Furthermore, our method can obtain any value (positive, zero, or negative) as the final result from an initial value of zero.

Note 8. For clarity, we do not use matrix algebra in this section.

Note 9. Note that the numerator and denominator on the right-hand side of the above equation contain this large value.

Note 10. While we can conceptually consider two other transformations: $f_{ti} = p_{ti}/y_t$ and $g_{ti} = q_{ti}/y_t$. these transformations lead to the same ideal index as that derived using Transformation-W. Therefore, the ideal log-change indices have only two formulae: the Vartia-Sato index and the Montgomery index.

Note 11. Letting $h = x_1/x_0$ as in Section 1 and using Eq. (D2), we have

$$L(x_1, x_0) = x_0 L(h, 1) = x_0 A\left(h^{1/2}, 1\right) A\left(h^{1/4}, 1\right) \ldots.$$

If $h = 1(\text{i.e.}, \Delta x = 0)$, all arithmetic means become unity. Thus, the limit (2) is obtained as follows:

$$\lim_{\Delta x \to 0} L(x_1, x_0) = x_0 \lim_{h \to 1} L(h, 1) = x_0.$$

Note 12. Letting $X = \left(x^{1/2} + 1\right)\left(x^{1/4} + 1\right)\left(x^{1/8} + 1\right)\left(x^{1/16} + 1\right) \ldots$, we can exploit the fine expansion of X as follows:

$$x^{1/2} + 1 = \left[x^{2/4} + 1\right], \left(x^{1/2} + 1\right)\left(x^{1/4} + 1\right) = \left[x^{2/4} + 1\right]\left(x^{1/4} + 1\right) = \left(x^{3/4} + x^{2/4} + x^{1/4} + 1\right)$$
$$= \left[x^{6/8} + x^{4/8} + x^{2/8} + 1\right],$$
$$\left[\left(x^{1/2} + 1\right)\left(x^{1/4} + 1\right)\right]\left(x^{1/8} + 1\right) = \left(x^{7/8} + x^{6/8} + x^{5/8} + x^{4/8} + x^{3/8} + x^{2/8} + x^{1/8} + 1\right) \ldots.$$