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Original Paper

Physical and Mathematical Model of Relaxation Filtration of Undergrounds Water

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Abstract

Earlier, a number of publications presented the results of an experimental study of the free oscillations of the piezometric level of groundwater (pressure in a liquid) in wells with natural frequencies. In this case, the oscillations were initiated by a pulsed action on the aquifer through disturbing wells. The same publications proposed a theoretical interpretation of the established phenomenon. However, a critical analysis of these theoretical constructions showed their incorrectness. Accordingly, a consistent physical and mathematical model of groundwater relaxation filtration is proposed in this work, which provides a theoretical basis for the description and interpretation of the free oscillations of the piezometric level in a disturbing well with natural frequencies. Thus, the proposed physical and mathematical model made it possible to conclude that it is impossible to cause resonance during the initiation of free oscillations of the piezometric groundwater level. An analysis of the experimental data based on the model showed that the implementation of methodological developments based on the theory of relaxation filtration is possible and expedient only on fundamentally different, in comparison with existing, technical and methodological approaches to the experimental study of wave propagation in aquifers.

Keywords

Free oscillations of the piezometric level (pressure), natural frequencies, theory

1. Introduction

As noted earlier in [21], in 1966, an article by J. D. Bredehoft, G.G. Cooper and I.S. Papadopoulos [10]. It based on analog modeling of groundwater (GW) filtration investigated inertial effects in aquifers. The article shows that one of the likely results of the manifestation of such effects may be the occurrence of waves in aquifers, which determine the free oscillations of the piezometric level of PV in wells with natural frequencies.

In the middle of the 70s, the articles by I. Krauss were published in German [30, 31], in which the results of field experimental studies of some problems of GW dynamics were presented. Thus, in the course of the experiments, free damped oscillations of the piezometric GW level were recorded in disturbing wells after pulsed excitation of pressure aquifers. In the same papers [30, 31], the first theoretical interpretation of the observed wave effect was proposed.

Subsequently, in a number of publications (see, for example, [8, 25, 26, 34]), the results of a theoretical interpretation of the process of free pressure oscillations p (or piezometric pressure H) in fluids in aquifers after their pulsed excitation, including, based on the model proposed in the works of I. Krauss [30, 31]. Later, in the late 80s, the works of S.F. Grigorenko [8, 25], devoted to the same problem. Their thesis was the PhD thesis [26], presented by S.F. Grigorenko to protect in 1992. It should be noted that in his theoretical constructions he used the physical and mathematical model of the GW movement, in principle, no different from the one proposed by I. Krauss.

The need to develop such a theory was dictated by a number of considerations. The main one is that such a development promised the creation of a fundamentally new experimental base in hydrogeology; interest in the problem of wave processes in aquifers has already been realized by the emergence of methods for conducting experimental filtration testing (EFT) of water-bearing sediments and the interpretation of the results of these experiments [8, 25, 26, 30, 31, 34].

At the same time, many aspects of the above methods are poorly substantiated theoretically, but for a number of positions they are simply incorrect. Accordingly, in our work [21], a critical analysis of the main principles of the physical and mathematical model of GW filtration used in the papers by I. Krauss and S.F. Grigorenko, who showed that the theoretical constructions used in these works are simply incorrect. This was the impetus for the development of the theory of relaxation filtration of GW, which adequately represents the experimentally established phenomenon. This theory was developed by the author (see [12-14, 16, 20, 22, 23, 50]) and is presented in this paper.

2. Method

2.1 Physical and Mathematical Model of relaxation Filtration of underground Water

Let us consider a physical and mathematical model of the movement of GW, providing, among other things, a description of free oscillations of a piezometric level. The justification and a detailed exposition of it are given in [12-14, 16, 20, 22, 23, 50], according to which the present work will be constructed. The complete formulation of this model consists in compiling, on the basis of the continuity equations, the state and the filtering law, of differential equations for the distribution of pressure in a fluid or head [18, 20]; the latter are related by the obvious correlation

$$p = H\rho g \tag{1}$$

where ρ is the density of the liquid (water); g-gravitational acceleration. Let us assume that the fluid (GW) moves in an isolated homogeneous and isotropic layer.

It is known from theoretical hydromechanics (this was also emphasized in [12-14, 16, 20, 22, 23, 50]) that free oscillations are determined by monochromatic standing waves [35]. In this connection, we recall two obvious propositions that form the basis for the concepts of such waves – standing waves form in a limited volume of a conducting medium, and their description and analysis should be performed on the basis of hyperbolic-type filtration equations (wave equations). These provisions, however, one way or another, were ignored in the theoretical justification of the above-mentioned methods [20]. Let us consider in more detail the physical prerequisites for the implementation of these provisions.

In the section, an isolated aquifer is conditionally distinguished by impermeable solid horizontal surfaces—a sole and a roof. The closeness of the space in which the waves propagate, in terms of the horizon, which has a large length, is due to the structurization of the filtering liquid, realizing the nonlinearity of the filtration law at small pressure gradients, in particular, the existence of the initial (limit) gradient in the liquid (modulus of the limiting gradient pressure is equal to G_p, and the hydraulic gradient—*G*). The nonlinearity of this kind of filtration law has been established by numerous experimental studies (see, for example, a brief review of such papers in [15, 18-20]). Vertical surfaces on which condition $\left|grad\ p\right| = G_p$ (or $\left|grad\ H\right| = G$) is fulfilled in the marginal parts of the perturbation region limit the perturbation region in plan. In the outer part of the aquifer, the fluid (GW) behaves like a solid [12, 20], in the disturbance region the structure of the fluid is completely destroyed, here the fluid behaves like Newtonian [9, 17, 18, 41].

It should be emphasized that the experimental registration of free oscillations with stable frequencies indicates a stable spatial position of the outer boundary of the perturbation region, and, therefore, the most reasonable from this point of view is the filtering law of the form [6, 7, 12, 16, 20] recorded for pressure in liquid

$$\vec{v} = -\frac{k}{\eta}\left(grad\ p - G_p\ \frac{grad\ p}{\left|grad\ p\right|} \right),$$

where, as before, *v* is the modulus of the filtering velocity vector; *k* is the permeability of the conducting (porous) medium; η is the viscosity of the fluid. The barodiffusion transfer into which the filtration at this boundary is transformed, by virtue of the incommensurably lower diffusion rate in comparison with the filtration rate, does not have any noticeable effect on the spatial position of the boundary [12, 20].

When pulsed excitation of the aquifer should occur, at least one traveling cylindrical wave. At its leading edge, the pressure gradient far exceeds the limit, however, as the wave advances, it flattens out, and at some distance *R* from the axis of the disturbing well, the pressure gradient at the wave front becomes equal to the limit, its advancement stops, and the traveling wave is reflected inside the disturbed area.

In the course of the movement of such a wave, the structure of the liquid is completely destroyed. Its restoration occurs through diffusion [5]—very slowly compared with destruction, therefore, it makes no sense to consider the conditions for the influence of structure recovery on the patterns of oscillations of the piezometric level for a short-term disturbance. Accordingly, the law of filtration of the fluid (in the inertialess motion) in the perturbed region becomes linear.

The basis of the wave filtration equation is usually based on the concepts of relaxation phenomena in liquids [36, 37] (Note 1). Maxwell on the basis of such ideas put forward the position of the absence of fundamental differences in the mechanical properties of liquids and solids. With a short exposure to fluids, they are elastically deformed; after the cessation of deformation, shear stresses remain in them, decaying with time. If the period $\frac{1}{\omega}$ of the change in the external (changing with frequency ω) force is large compared with the duration of the decay of the voltages τ_0, $\omega\tau_0 << 1$, then the liquid will behave like a regular viscous liquid. At sufficiently high frequencies, when $\omega\tau_0 >> 1$, the liquid will behave like an amorphous solid [36].

Let us take the simplest assumption about the law of attenuation of internal stresses (after the cessation of motion), we will assume that it occurs by a simple exponential dependence (which, by the way, has a solid theoretical and experimental justification [47]). The latter corresponds to a similar dependence for shear deformations, which takes the form $\frac{d\vec{v}}{dt} = -\frac{1}{\tau_0}\vec{v}$ in terms of a filtration phenomenological model, where *t* is time.

It is not difficult to see that the equation

$$\vec{v} + \tau_0 \frac{d\vec{v}}{dt} = -\frac{k}{\eta}\, grad\ p \tag{2}$$

eads to correct results in both limiting cases of short-term and prolonged exposure to a liquid in a porous medium, and therefore can serve as an interpolation equation for intermediate cases. So, for periodic motion, when $grad\ p$ and $\vec{v}$ depend on time by means of the factor $e^{-i\omega t}$ [36], from (2) we have

$$\vec{v} = -\frac{1}{(1 - i\,\omega\tau_0)}\frac{k}{\eta}\, grad\ p\,.$$

When $\omega\tau_0 << 1$, this formula gives

$$\vec{v} = -\frac{k}{\eta}\, grad\ p \tag{3}$$

i.e., we obtain the filtration law of a viscous fluid (Darcy's law), and with $\omega\tau_0 >> 1$

$$\frac{k}{\eta}\, grad\ p = i\,\omega\tau_0\,\vec{v} = -\tau_0\frac{d\vec{v}}{dt}\ -$$

an expression similar to the expression for tangential stresses in a solid, written in terms of the filtration model.

Equation (2) is similar to the equation for the motion of an viscoelastic fluid [40].

In order to close the system of equations of the filtration model, we use the continuity equation

$$\frac{\partial n\rho}{\partial t}+div\left(\rho\vec{v}\right)=0$$

(n is the porosity of the conducting medium; ρ is the fluid density, as before), and also by the fact that the properties of the fluid, as well as the porosity and permeability of the conducting medium, are functions of pressure. Assuming weak compressibility of the fluid and porous medium [6, 7]

$$\frac{\partial\rho}{\partial p}=\frac{\rho_0}{E_\rho},\quad \frac{\partial n}{\partial p}=\frac{n_0}{E_n},\quad \frac{\partial}{\partial p}\left(\frac{k}{\eta}\right)=\frac{k_0}{\eta_0\,E_k} \tag{4}$$

where n_0, k_0, ρ_0, η_0—respectively, the porosity and permeability of the medium, the density and viscosity of the fluid at pressure p_0 (unperturbed horizon); E_ρ, E_n and E_k—modules of volumetric compression of water and porous medium.

Substituting into the continuity equation the equations of state from (4) and the filtration law (2) lead to a relation similar to the wave equation for heat conduction [37],

$$\frac{\partial p}{\partial t}+\tau_0\,\frac{\partial^2 p}{\partial t^2}=\chi\,\Delta\,p,\quad \chi=\frac{k_0}{\eta_0}\left(\frac{n_0}{E_\rho}+\frac{1}{E_n}\right)^{-1} \tag{5}$$

Here χ is the piezoconductivity of sediments; Δ is the Laplace operator. The remaining notation is the same.

To complete the formulation of the problem, it is necessary to determine the initial and boundary conditions. Consider the problem of the propagation of waves with their own frequencies in an aquifer after pulsing it through a disturbing well, for example, according to the method proposed in [8, 25, 26, 30, 31]. According to it, the piezometric level in a well with a sealed wellhead was squeezed out with compressed air, then the well was maintained at a constant gas pressure until the dynamic water level in it (with maximum pressure drop p^0) and pressure in the disturbed region stabilized. At time $t=0$, the wellhead is depressurized, and the excess air pressure in it is abruptly dropped, with the result that waves with natural frequencies are initiated in the aquifer, the manifestation of which in the disturbing well are free oscillations of the piezometric level GW; we confine ourselves here to the consideration of the steady state level fluctuations. At the outer boundary of the perturbation region, at a distance R from its center, the conditions $\left.\frac{\partial p}{\partial r}\right|_{r=R-0}=0$ and, respectively, $v=0$ are satisfied [12, 20].

We will neglect for the simplicity and visibility of the solution the capacity of the well. We transform equation (5) for cylindrical waves as follows [12, 20]:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} - \frac{\tau_0}{\chi}\frac{\partial^2 p}{\partial t^2} - \frac{1}{\chi}\frac{\partial p}{\partial t} = 0\,. \quad (6)$$

The solution of the problem will be sought in the form [12, 20, 35]

$$p(r,t) = Ae^{-(\beta+i\omega)t} \cdot f(r) \quad (7)$$

where A is some constant; p is the pressure change relative to the initial p^0; β—oscillation damping factor; $f(r)$ is some function of the coordinate.

Substitution (7) in (6) gives for function $f(r)$

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \kappa^2 f = 0, \quad \kappa^2 = \frac{\tau_0}{\chi}\left(\omega^2 - \beta^2 + \frac{\beta}{\tau_0}\right) + \frac{1}{\chi}(1 - 2\beta\tau_0)\, i\,\omega \quad (8)$$

his is the zero order Bessel function equation. In a standing cylindrical wave, p must remain finite at $r_0 \to 0$ (r_0 is the well radius); The corresponding solution is the $J_0(\kappa r)$—Bessel function of the first kind of order zero [1, 35].

Thus, (7) can be represented as

$$p(r,t) = p^0\, e^{-(\beta+i\omega)t} \cdot J_0(\kappa r).$$

We restrict ourselves to the real part, we have [12, 20]

$$p(r,t) = p^0\, e^{-\beta t} \cdot \cos(\omega t \pm \phi) \cdot J_0(\kappa r) \quad (9)$$

where ϕ is the phase angle.

From the asymptotic representation of the Bessel functions, it follows that the argument in $J_0(\kappa r)$ must be a real number. Therefore, in the second equation (8) it is necessary to put $\operatorname{Im}\kappa^2 = 0$ and $2\beta\tau_0 - 1 = 0$, whence, in particular, the relation $\beta = \frac{1}{2\tau_0}$ follows. Accordingly, instead of the second equation in (8) we get [12, 20]

$$\kappa^2 = \frac{\tau_0(\omega^2 - \beta^2) + \beta}{\chi} \quad (10)$$

The oscillatory system under consideration has an infinite number of natural oscillation frequencies, it is only necessary that an integer number of half-waves fit within the radius of the disturbance region: $\kappa R = \frac{\lambda_N}{2} N$ (λ_N is the wavelength, N = 1, 2, ...). The available boundary condition provides an estimate of these frequencies. Take

$$\left.\frac{dp}{dr}\right|_{r=R} = -p^0 e^{-\beta t} \cdot \cos(\omega t \pm \phi) \cdot \kappa J_1(\kappa R) = 0,$$

whence $J_1(\kappa R) = 0$. The last equation has a countable set of roots $j_{1,N}$:

$$\kappa_N R = j_{1,N}.$$

Then, taking into account (10), we obtain [12, 20]

$$\omega_N = \sqrt{\frac{j_{1,N}^2 \cdot c^2}{R^2} - \beta^2} = \sqrt{\omega_{0N}^2 - \beta^2} \tag{11}$$

Here, $c = \sqrt{\frac{\chi}{\tau_0}}$ is the wave propagation velocity [37]; $\frac{j_{1,N} \cdot c}{R} = \omega_{0N}$—natural frequencies of oscillations in the absence of friction forces (resonant frequencies).

Expression (9) describes the free pressure oscillations in the liquid, the amplitude of which decreases throughout the experiment exponentially. The fuzziness and pressure nodes are entirely determined by the maxima, minima, and zeros of function $J_0(\kappa r)$. Obviously, the solution obtained exists for all values of β, ω_N, and τ_0 that satisfy conditions $2\beta\tau_0 - 1 = 0$ and $\tau_0 \omega_N^2 > \beta(\beta\tau_0 - 1)$.

The model of relaxation filtration was constructed by us under the assumption of weak damping, when $\beta << \omega_0$, i.e. The registered natural frequency is only slightly lower than the resonant one. With strong damping, when $\omega_0^2 < \beta^2$, the value of ω becomes imaginary. In this case, a complex solution that has a physical meaning does not exist; two independent solutions [45] must correspond to different signs in expression (11), but then we arrive at a problem for which there are no non-trivial solutions. Therefore, we confine ourselves to only one of the independent solutions, and the sign in front of $\sqrt{\omega_0^2 - \beta^2}$ is chosen from the condition of wave radiation in the direction from the well.

On the other hand, the function $f(r)$ ~ $J_0(\kappa r)$, where $\kappa^2 = \frac{\upsilon - \upsilon^2 \tau_0}{\chi}$ and $\upsilon = \beta - \sqrt{\beta^2 - \omega_0^2} > 0$, obtained from equation (8), exists only in the range of values of ω_0 and β, for which $\upsilon \tau_0 < 1$, i.e., for very small values of υ. Then the solution shows that immediately after the pulse impact on the reservoir in the area of disturbance, the pressure distribution becomes similar to the standing wave; further, the pressure deviation from the initial one at each point of the region, without making oscillations, naturally decreases.

In the case when $\upsilon\tau_0 > 1$, instead of (8) we get [12, 20]

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \kappa^2 f = 0, \quad \kappa^2 = \frac{\upsilon^2\tau_0 - \upsilon}{\chi}.$$

The solution of the last equation is the function [1, 48]

$$f(r) = C_1 I_0(\kappa r) + C_2 K_0(\kappa r) \quad (12)$$

Here, C_1 and C_2 are constant; $I_0(\kappa r)$ and $K_0(\kappa r)$ are modified Bessel functions of zero order, respectively, of the first and second kind [1].

Since for $r \to \infty,\ I_0(\kappa r) \to \infty$, in (12) we should put $C_1 = 0$. From the condition for $t = 0$ and $r = r_0$ we find the constants AC_2, whence the solution of the problem takes the form [12, 20]

$$p(r,t) = A^* e^{-\upsilon t} \cdot K_0(\kappa r), \quad A^* = \frac{p^0}{K_0(\kappa r_0)} \quad (13)$$

showing—the process of pressure recovery in the aquifer is no longer determined by standing waves.

It should be noted that the last expression almost completely coincides with the solution of I. Krauss [30]. The difference lies in the absence in (13) of a factor that determines the pressure oscillations in the fluid (or piezometric head). In addition, the function $K_0(\kappa r)$ in (13) exists only in the perturbation region, on the external contour of which condition $|grad\ p| = G_p$ is satisfied, i.e. in region $r_0 \le r \le R$, whereas according to the solution of I. Krauss, the perturbation instantaneously (in time) extends to infinity, as is assumed in solutions of the filter equation of parabolic type.

A completely analogous solution is obtained in case $\beta = \omega_0$, it is enough to replace υ with β.

The available experimental data from [8, 25, 26, 30, 31, 34] (see, Fig. 1) are in excellent agreement with the physical and mathematical model obtained for the relaxation filtration of GW. At the same time, the possibility of constructing on the basis of this model techniques EFT of aquifers and interpretation of the results of such testing requires a thorough independent study and will be discussed below.

A similar picture should be observed in the case of fluid movement with central symmetry (impulsive perturbation of a high-power aquifer through an imperfect hole that is not perfect in terms of opening—a point piezometer). Moreover, the growth of the area of the frontal surface of a spherical wave occurs in direct proportion to the increase in the square of the radius to its front [20, 49].

It is easy to see that equation (5) for the case of filtration with central symmetry is converted to

$$\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} - \frac{\tau_0}{\chi}\frac{\partial^2 p}{\partial t^2} - \frac{1}{\chi}\frac{\partial p}{\partial t} = 0 \quad (14)$$

Its solution under the same boundary conditions as in the case of filtration with axial symmetry is the expression for a monochromatic standing spherical wave of the form [35]

$$p = A^* e^{-\beta t} \cdot \cos\left(\omega t \pm \phi\right) \cdot \frac{\sin \kappa r}{r}, \quad A^* = \frac{p^0 r_0}{\sin \kappa r_0},$$

where all designations are the same.

The natural frequencies of this oscillatory system are determined by the same expression (11). The above analysis is applicable to the above solution, and for $\upsilon \tau_0 < 1$, the solution of equation (14) takes the form

$$p = A^* e^{-mt} \cdot \frac{\sin \kappa r}{r}, \quad A^* = \frac{p^0 r_0}{\sin \kappa r_0},$$

showing that, as in the case of fluid movement with axial symmetry, immediately after the impact on the aquifer in the disturbance region, the pressure distribution becomes similar to the standing wave. Further, the pressure devia-

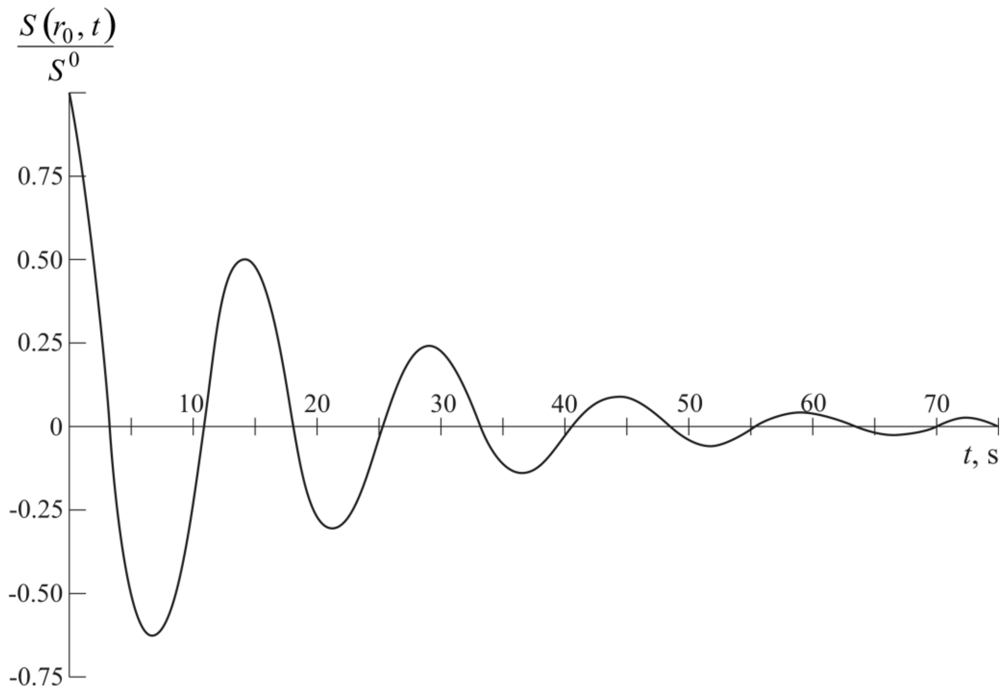

Figure 1. The Indicator Curve for Tracking Fluctuations of the Piezometric Level Decreases in the Well

(According to I. Krauss [20, 21]). Here $S(r_0, t)$ Is the Lowering of the Level in the Perturbing Well; S^0—initial (Pressurized by Compressed Air) Lowering of the Level at the Time of the Start of the Disturbance

tion from the initial one at each point of the region, without making oscillations, naturally decreases. When $\upsilon\tau_0 > 1$, the solution to equation (14) is

$$p = A^* e^{-\upsilon t} \cdot \frac{e^{-\kappa r}}{r}, \quad A^* = \frac{p^0 r_0}{e^{-\kappa r_0}}, \quad r_0 \le r \le R.$$

We now turn to the analysis of the distribution of plane waves in the pressure aquifer. Let us suppose that, as in the previous cases, after a pulsed perturbation, oscillations of the piezometric level GW with eigenfrequencies also occur here (Note 2). In the same way, as in [12, 20], we consider that the motion of a GW after a pulsed disturbance of an aquifer is described by a differential filtering equation in partial derivatives of hyperbolic type (5), converted for a plane flow to the form [20, 49]

$$\frac{\partial^2 p}{\partial x^2} - \frac{\tau_0}{\chi}\frac{\partial^2 p}{\partial t^2} - \frac{1}{\chi}\frac{\partial p}{\partial t} = 0 \qquad (15)$$

Here all the designations are the same.

To complete the formulation of the problem, we accept, as before, that the maximum pressure differential in the drain in a pulsed perturbation is set equal to p^0.

We restrict ourselves, as before, to the consideration of the steady-state process of pressure oscillations in a liquid; the solution of the problem will be sought in the form [35]

$p(x, t) = Ae^{-(\beta + i\omega)t} \cdot f(x)$, (16)

where $p(x, t)$ is the pressure change in the fluid at the point of the aquifer with the *x* coordinate (measured from the vertical wall of the drain) at time *t* from the onset of the disturbance relative to the initial p_0. All other designations are the same.

Substituting (16) into (15) gives for function $f(x)$

$$\frac{d^2 f}{dx^2} + \kappa^2 f = 0, \quad \kappa^2 = \frac{\tau_0}{\chi}\left(\omega^2 - \beta^2 + \frac{\beta}{\tau_0}\right) + \frac{1}{\chi}(1 - 2\beta\tau_0)i\omega \qquad (17)$$

Attention should be paid to the fact that the expression for κ^2 completely coincides with the similar expression for the cases of fluid motion with axial and central symmetry.

The solution of equation (17) are cylindrical Bessel functions of half-integer order of the first and second kind [1]:

$$f(x) = C_1\sqrt{x} \cdot J_{\frac{1}{2}}(\kappa x) + C_2\sqrt{x} \cdot Y_{\frac{1}{2}}(\kappa x) \qquad (18)$$

The values of the functions $J_{\frac{1}{2}}(z)$ and $Y_{\frac{1}{2}}(z)$ in (18) were tabulated in [20] using the existing tables of spherical Bessel functions of integer (zero) order of the first and second kind from [1], their graphs are shown in Figure 2.

By analogy with a cylindrical standing wave, we assume that in a flat wave *p* should remain finite at $x=0$. The function $Y_{\frac{1}{2}}(\kappa x)\to\infty$ at $x\to 0$ (see Fig. 2). Therefore, in (18) we should take $C_2=0$. Henceforth, we get [20, 49]

$$f(x)=C_1\sqrt{x}\cdot J_{\frac{1}{2}}(\kappa x).$$

We investigate the obtained solution without further specifying its appearance. As can be seen from Fig. 2, with $x=0$, function $J_{\frac{1}{2}}(\kappa x)=0$. This means that immediately after the impulsive perturbation of the aquifer, the pressure in the drainage instantly returns to its original state (the pressure here becomes equal to p_0) and then remains constant, i.e., there is no pressure fluctuation (or piezometric head) with natural frequencies [20].

The latter, firstly, contradicts the initial assumption about the presence of such oscillations, and, secondly, shows the absence of standing waves in the aquifer. In fact, as noted earlier, the areas of the frontal surface of traveling cylindrical and spherical waves grow rapidly as they advance, which leads to a flattening of the waves and a decrease in the pressure gradient on their front. In a flat traveling wave, the frontal surface area remains constant, and the wave does not flatten out (Note 3). Accordingly, there is no necessary condition for the formation of a solid impermeable surface on the external contour of the perturbation region, $|grad\ p|=G_p$, and, therefore, the traveling wave is not reflected inside the disturbed region, and flat standing waves are not formed. Naturally, therefore, pressure oscillations (or piezometric pressure) with Eigen frequencies in a flat flow should not occur (as follows from the above formal calculations) [20, 49].

The above solutions, as well as solutions from the works of I. Krauss and S.F. Grigorenko [8, 26, 30, 31], in fact, are kinematic, and, therefore, they do not explicitly express the relationship between such pressure fluctuation characteristics (or piezometric level) as the attenuation coefficient β and the duration of shear stress damping τ_0, with the parameters of the medium in which standing waves are formed, i.e. with parameters of the aquifer. And meanwhile, it is obvious that both the duration of shear stress damping τ_0 and the coefficient associated with it by the ratio $\beta=\dfrac{1}{2\tau_0}$ must be determined by the filtration and geometrical properties of the conducting medium. Let us try to identify this conditionality using a different, in comparison with that used above, approach to substantiating the type of filtering law (2) in the perturbed region proposed in [27] with references to numerous previous studies, in other words, we will try to build a hydrodynamic model of the piezometric oscillations level

In accordance with this approach, we select a certain elementary volume of a statistically homogeneous porous medium W_P, physically at the same time much larger than the sizes of the pores and grains of the medium; only for a volume covering a large number of them are the average characteristics sufficiently representative [17, 18, 42, 43]. The pore volume by which the fluid is filtered is equal to $W_G = n_D W_P$ in this element (here n_D is the dynamic, active porosity of the conducting medium). Consider the forces acting on the fluid.

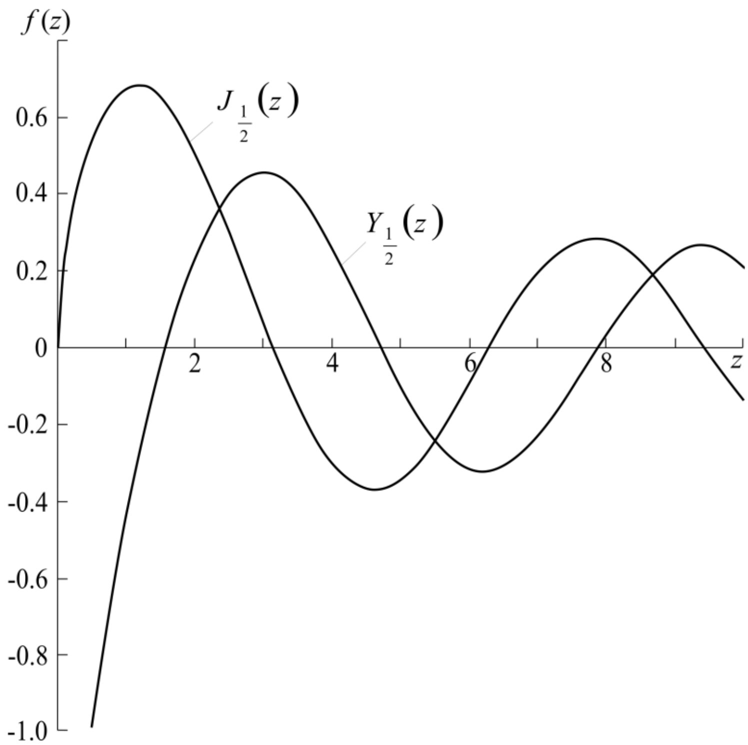

Figure 2. Graphs of Bessel Functions of Half-Integral Order

The surface of the selected volume of fluid in accordance with the third law of Newton is affected by pressure from adjacent layers of water and a solid skeleton of a porous medium in the direction of the internal normal to the surface *S* of the volume W_G. Accordingly, the pressure is $\oint_S p\,dS$. Since the surface *S* is closed and the pressure function in the fluid is continuous together with their first derivatives, the Gauss transformation is applicable to the last expression [6, 29]:

$$\oint_S p\,dS = -\int_{W_G} \nabla p\,dW_G ,$$

where ∇ is the Hamilton operator.

Filtration laminar fluid, as has been repeatedly emphasized, is experiencing resistance proportional to the first degree of filtration rate. Assuming the fluid is purely viscous, and using the Darcy law (for a perturbed region), we write the volume force of resistance (friction) as

$$\int_{W_G} \frac{\eta}{k} \vec{v}\,dW_G .$$

All designations here are the same.

According to the Dalamber principle, the sum of the forces acting on the system of material points is equal to the force of inertia. The latter in volume W_G is [27]

$$\int_{W_G} \rho \frac{d\vec{u}}{dt} dW_G$$

(here *u* is the modulus of the velocity vector (not filtering) of the fluid). Since $n_D \vec{u} = \vec{v}$, then

$$\int_{W_G} \rho \frac{d\vec{u}}{dt} dW_G = \int_{W_G} \frac{1}{n_D} \rho \frac{d\vec{v}}{dt} dW_G .$$

From here we get—

$$\int_{W_G} \nabla p \, dW_G - \int_{W_G} \frac{\eta}{k} \vec{v} \, dW_G = \int_{W_G} \frac{1}{n_D} \rho \frac{d\vec{v}}{dt} dW_G \quad (19)$$

Since the taken volume W_G is arbitrary, it follows from (19)

$$\vec{v} = -\frac{k}{\eta}\left(\nabla p + \frac{1}{n_D} \rho \frac{d\vec{v}}{dt} \right)$$

or, after simple transformations [20], —

$$\vec{v} = -\frac{k}{\eta} \nabla p - \frac{k\rho}{\eta n_D} \frac{d\vec{v}}{dt} .$$

The last equation completely coincides with (2), however, in it the duration of shear stress attenuation is expressed in terms of the parameters of the medium and the filtering fluid:

$$\tau_0 = \frac{k\rho}{\eta n_D},$$

or, if we go from the permeability of the medium to the filtration coefficient *K* and from the pressure in the liquid to piezometric heads (see equation (1)) [20], —

$$\tau_0 = \frac{K}{g\, n_D} . \quad (20)$$

All designations here are the same. It is obvious that in (20) the value of the filtration coefficient *K* must be specified in m/s.

Accordingly, the attenuation coefficient of oscillations β gets a quite clear physical interpretation, according to which its higher values are characteristic for media with lower permeability and vice versa [20]:

$$\beta = \frac{\eta}{2k} \frac{n_D}{\rho}, \text{ or } \beta = \frac{g n_D}{2K} \quad (21)$$

Let us try to evaluate the applicability and correctness of such an interpretation of the parameters τ_0 and β. This can be done by identifying how equations (20) and (21) correspond to the available experimental data on filtration and migration parameters.

Considering the characteristic value of $\tau_0 \sim N \cdot 10^0$ c in equation (20) (this order of magnitude follows from the available experimental data given in [30, 31]), as well as g ~ 10 m/s^2, we get that [20]

$$n_D \cong (0{,}1-1)\,K\,. \quad (22)$$

By setting in (22) the control values of the filtration coefficient in the range from 10 to 200 m/day (or $K \sim 1.2 \cdot 10^{-4}$-$2.3 \cdot 10^{-3}$ m/s, respectively), it can be shown that n_D in this case should be about $1.2 \cdot 10^{-5}$-$2.3 \cdot 10^{-3}$.

The values of the dynamic, active porosity of the conductive medium as a whole correspond to the characteristic values of active fracturing established using a fundamentally different methodological base —Experimental Migration Testing (EMT) of the fractured aquifers—and components $5 \cdot 10^{-4}$-10^{-2} [2]. It is obvious at the same time that such active fracturing characterizes only the capacity of cracks without taking into account water exchange with porous blocks.

In principle, the situation is different with the active porosity of loose (sandy) sediments, which in the order of magnitude reaches 10^{-1}. If we start from the interval of characteristic values of active porosity n_D, presented in [2], it should be recognized that relaxation processes should manifest themselves in environments for which the K values are $5 \cdot 10^{-3}$-1 m/s or 430-86400 m/day. The absurdity of the obtained interval of probable values of K is obvious; conductive porous media with such filter coefficients are unlikely to actually exist in nature. At the same time, the results of experimental studies are reported in [31], according to which free oscillations of the level of the water source are recorded in an aquifer with a filtration coefficient of about 160 m/day, and according to the analog modeling data, fluctuations of the level with natural frequencies are possible in conducting media with $K \sim 40$ m/day [10].

The indicated contradiction is an obvious manifestation of the incomplete correspondence of the concepts of dynamic, active porosity and actual flow rate, movement (not filtration) of the GW, whose modulus is equal to u, to the concepts of the phenomenological filtration model (see physical and filtration phenomenological models for [17, 18]). This discrepancy is also largely determined by the methodological basis, on the basis of which the experimental determination of these parameters is carried out; as mentioned earlier, they are determined by the methods of EMT aquifer systems.

So, in particular, as a result of the EMT of the aquifer (as a rule, according to the transit time with a natural or initiated perturbation of the flow of GW of some label between the designated spatial boundaries of the test horizon), the average actual water velocity and the dynamic porosity of the conducting medium are calculated. At the same time, in accordance with the ideas of the filtration phenomenological model, this experimentally established velocity u is effective (that is, apparent), since it is formally related to the length of the tested interval of the horizon. Accordingly, the dynamic porosity is estimated as the ratio of the effective actual flow rate of water to the effective filtration rate of water *v*, i.e., is also an effective (apparent) value.

Thus, the fundamental difference between u and v lies only in the fact that the filtration rate *v* implies the movement of water over the entire cross section of the elementary volume of the conducting medium, normal to the direction of flow of the GW, without taking into account its part occupied by the solid skeleton of the porous medium, whereas the actual flow velocity *u*—only in part of this section, determined by the very dynamic porosity.

The idea of the actual water flow rate as a certain effective value that is not a parameter representing a real physical characteristic of the water flow is evidenced, for example, by the fact that when estimating a given interval length of the aquifer tested by the EMT methods, the correction (factor) pore curvature. Thus, in [28, 44], this amendment is proposed to be considered constant and equal to 1.6. Already this alone determines a noticeable excess of the adjusted actual speed of water movement over a certain experimentally—at least 60% relative to the latter.

Another important feature of the actual water velocity (and, accordingly, dynamic porosity) determined by the EMT methods is the fact that these methods are oriented to the inertial motion of the GW. In other words, it is assumed that with such testing, water (as well as the dissolved label contained in it) flows through all open non-dead pores (or cracks) of the conducting medium; water exchange is also carried out with dead-end pores (or with porous blocks of fractured-porous medium). In this case, a prolonged exposure to water in a conducting porous medium in the course of such an EMT causes the destruction of the structure of water interacting with the solid pore surface in almost all of these pores.

A short-term, impulsive, alternating effect on an aquifer during the initiation and propagation of pressure waves (or piezometric pressure) with natural frequencies in it determines the destruction of the water structure only in the largest interconnecting pores, the number of which is extremely limited even in highly permeable environments. An essential condition for the implementation of such an impact is the excess of the characteristic pore diameter (or crack opening) over two times the thickness of a layer of strongly bound water adjacent to the pore walls or cracks, postulated in a three-layer model of water structure, considered and adapted for analyzing the process of water movement in pores and cracks, for example in [9, 41]. Accordingly, the effective value of the dynamic porosity in equation (19) in this case should be many times less than that determined by the EMT methods.

A similar approach to accounting for the inertial component of resistance was proposed in the work of V.S. Golubeva [24].

Thus, in justifying the physical and mathematical model of relaxation filtration of GW, effects appear that cannot be determined in terms of only the filtration phenomenological model. Thus, in the filtration model, the idea of the inertia of fluid motion is not so obvious; as a rule, inertial forces are taken into account by the type of the filtering law equation used in a particular physical and mathematical filtration model [6, 7, 18]. In the model under consideration, the effect is manifested, explicitly defined only by the terms of the phenomenological physical model of fluid (water) movement in pores or cracks.

Accordingly, in order to reconcile this effect with the conceptual basis of the filtration model, it is necessary to introduce some parameter P in the expression of inertial force [20], which should select from the total value of active porosity n_D that part of it in which the inertia forces actually act. Obviously, one of the components of such a parameter can apparently be considered the above-mentioned correction to the modulus of the velocity vector of the GW *u* for the curvature of the pores.

Equations (20) and (21) are converted respectively to the following form [20]:

$$\tau_0 = \frac{K}{g\mathrm{P}n_D}, \quad \beta = \frac{g\mathrm{P}n_D}{2K} \tag{23}$$

All designations here are the same. In the equations in (23), the dimensionless parameter P is in the order of magnitude $n\cdot(10^0\text{-}10^4)$, where *n* is a real non-negative number.·

Note that the minimum values of the dimensionless parameter $\mathrm{P} = n\cdot(10^0\text{-}10^4)$ presented here are related to fissure conductive media, and the maximum to porous ones.

As noted earlier, the available experimental data from [8, 25, 26, 30, 31, 34] are in excellent agreement with the constructed physical and mathematical model of relaxation filtration of GW. In Fig. 4 shows the results of tracing the free level fluctuations in a disturbing well, traversed in dolomites in the Regensburg region (Germany), after pulsed excitation of an aquifer taken from [31]. Processing them in accordance with the model described here gives the following characteristics of the level fluctuations in the well: $\beta_W \cong 0.054$ s^{-1}; $\omega_W \cong 0.448$ s^{-1}; $\tau_0 \cong 9.26$ s (index "*W*" here denotes the parameters set for the perturbing well).

In conclusion, it is necessary to point out another important circumstance. The presented physical and mathematical model of relaxation filtration of the GW was obtained under the assumption of the formation of standing waves with its own frequencies in pressure aquifers. For example, experimentally, the value of $\tau_0 \sim 10$ s [31] and the characteristic piezoconductivity of such horizons $\chi \cong 10^5\text{-}10^6$ m^2/day (or, respectively, 1.16-11.6 m^2/s), the wave propagation velocity in them, as follows from the explanation to expression (11), should be in the order of magnitudes c ≅ 0.34-1.1 m/s.

From the above, for example, in Fig. 1 experimental graph of pressure fluctuations (head or piezometric level) follows—in the well after the disturbance of the aquifer the wave process becomes stationary after the first half period of oscillations, i.e., after, for example, 8-10 s after the start of the experiment with a period of oscillations of 15-20 s. During this period of time, the radiated traveling cylindrical vacuum wave should reach the solid boundary of the disturbed region at a distance *R* from the disturbing well, reflect from this boundary and return to the well. Accordingly, in the well at the moment of arrival of the reflected wave, the maximum pressure in the fluid (or piezometric head) is fixed, and then a stationary process of damped pressure fluctuations or piezometric head (including, also, in the disturbing well) develops.

With such a probable range of values of the velocity of propagation of waves c and the duration of stabilization of the process with the same 8-10 s, the distance *R* to the border can be only ~ 1.36-5.5 m. It is clear that even with a small initial p^0 (and corresponding the initial change in the piezometric head H^0, specified, for example, in the experiments of I. Krauss equal to 10-15 cm [30, 31]) $\left.\frac{\partial H}{\partial r}\right|_{r=R+0} > G$; herewith, in the order of magnitudes $G \cong n\cdot(10^{-4}\text{-}10^{-5})$ [19, 44].

In other words, even strictly formally, with a probable *R* value even in the first tens of m (for example, 30-40 m), to stabilize the process, it will take at least 15-60 s, i.e., it will take time that is clearly longer than the first half-period of pressure oscillations (or piezometric head) and in order of magnitude commensurate with the duration of the whole experiment.

Obvious absurdity of the result. Therefore, let us try to estimate the probable speeds of wave propagation in aquifers, based on available experimental data and a detailed presentation of the physics of wave formation and propagation in an aquifer.

We note here that the process under consideration is to a certain extent identical to the process of the propagation of shock waves (water hammer) in pressure pipelines [33, 52]. So, if the pressure at the beginning of the pipeline remains unchanged, then after the shock wave reaches the discharge from the quickly open valve in another section of the pipeline of this initial section of the pipe, the reverse movement of the shock wave starts at the same speed *c*, and this is a pressure increase wave. Upon reaching this shock wave section at the valve, the pressure here is made less than the initial pressure before the impact, after which the shock wave begins to move, but again the pressure decreases towards the beginning of the pipeline. The cycles of increase and decrease in pressure will alternate further at intervals equal to the double-shock time of the length of the pipeline from the valve to the beginning of the pipeline.

Thus, during a hydraulic shock, the fluid in the pipeline will oscillate, which, due to hydraulic resistance and viscosity, absorbing the initial energy of the fluid to overcome friction, will be damped.

As is known, the speed of wave propagation in water (including shock and vacuum waves) under constant conditions (temperature and pressure) is constant and corresponds to the speed of sound, i.e., $c \cong$ 1500 m/s. However, it is important to note that during hydraulic shocks in pipes, the speed of their passage is significantly affected by the pipe material [33, 52]. The latter circumstance causes a noticeable decrease in the speed of wave propagation relative to the speed of sound, up to about 900-1200 m/s.

Nevertheless, the probable values of the velocity of propagation of waves in pressure aquifers presented above are still incommensurable with the velocity of propagation of shock waves in water.

Let us now specify the scheme for conducting experiments from [8, 25, 26, 30, 31, 34], which determines the initial and boundary conditions for filtering GW in an aquifer when waves with natural frequencies are initiated in it. This scheme should determine the identity of wave propagation processes in an aquifer in accordance with experiments from [8, 25, 26, 30, 31, 34] and propagation of shock waves in pressure pipelines.

In accordance with this scheme, as already indicated, a column of water was squeezed out in each test well with a sealed mouth of compressed air (i.e., the piezometric level of the GW decreased), then the well stood up to a complete stabilization of this level at a certain constant level. At the same time, a maximum of air pressure was recorded in the well, but a minimum of piezometric head (or pressure in water). It is obvious that during the maturing of the well, the piezometric level of the GW near them was initially disturbed (elevated, as during injection of the GW into the aquifer) stabilized at the initial elevations. In other words, a decrease in the piezometric head (or pressure in the fluid) relative to the undisturbed head (or pressure) in the aquifer was observed in the test well (and only in it). This difference in pressure (or pressure) was fully compensated for by the overpressure of the air in the well above the GW level in it, respectively, the test well-aquifer system was brought to an equilibrium steady state.

At the time of depressurization of the wellhead, the excess air pressure over the water column in it quickly, almost instantaneously (in comparison with the processes occurring in the aquifer) was discharged to atmospheric due to incommensurably lower (almost two orders of magnitude) air viscosity in comparison with the viscosity of water. Correspondingly, in the test well, a free, not occupied by water instantly appeared, volume $V_0 = \pi r_0^2 H^0$, which very quickly began to fill with water from the filter zone of the well; Here a pressure-discharge area (or piezometric head) was formed, and this area began to move in all directions from the pilot well. In other words, at the time of depressurization of the mouth of an experimental well in the aquifer near it, a running cylindrical vacuum wave directed from the well was formed and detached from it.

As noted earlier, with an increase in the radius of the traveling cylindrical wave at its leading edge, the hydraulic gradient initially quite significant rapidly decreased. At a certain distance *R* from the axis of the test well, this gradient decreased to such an extent that the necessary condition $\left.\frac{\partial H}{\partial r}\right|_{r=R} \leq G$ was fulfilled. In such a case, this boundary appears to be some solid impermeable vertical cylindrical surface causing the probability of reflection of the traveling wave emitted from the test well back to the well. At the same time, as in the reflection of a shock wave (hydraulic shock) from the beginning of the pipeline, this is already a wave of increasing pressure (or piezometric head); The severity of the maximum pressure increase in the test well was enhanced by the cumulative effect of compression of the reflected cylindrical wave. The imposition of radiated and reflected traveling waves formed standing waves, and then cycles of increases and decreases in pressure (or piezometric head) alternated at intervals of time equal to the double-run time wave of distance *R*.

It is also important to note that, when an external solid reaches the radiated wave, the perturbation boundary on its inner surface, the previously valid boundary condition $\left.\frac{\partial H}{\partial r}\right|_{r=R} \leq G$, was instantly transformed into the boundary condition $\left.\frac{\partial p}{\partial r}\right|_{r=R-0} = 0$ (or $\left.\frac{\partial H}{\partial r}\right|_{r=R-0} = 0$) and $v = 0$ respectively indicated earlier [12, 20]. The latter just reflected the presence of a "solid" impenetrable vertical cylindrical surface at a distance *R* from the axis of the pilot well. Moreover, one of the known antinodes of the standing wave was formed on this surface (the second known antinode was formed on the axis of the perturbing well).

It is obvious that in reality the speed of wave propagation in aquifers should noticeably (at least by an order of magnitude) exceed those shown earlier with c ≅ 0.34-1.1 m/s. Therefore, the expression of the same dimensionless parameter P—

$$c = \sqrt{\frac{\chi \mathrm{P}}{\tau_0}} \tag{24}$$

All designations here are the same.

Accordingly, when the value of P for porous media, as noted earlier, is of the order of 10^2-10^4, the speed of propagation of waves c in (24) will be in the order of magnitudes $n\cdot(10^0$-$10^2)$ m/s. These wave propagation speeds show that the probable values of the distance *R* from the perturbing well to the solid outer boundary of the wave propagation region for the above-specified values of the oscillation period are in the order of 20-540 m. And already at such distances from the center of the disturbance the necessary boundary condition $\left.\frac{\partial H}{\partial r}\right|_{r=R} \le G$, causing the probability of reflection of the traveling wave back to the well.

It is clear that in a porous medium, just like in pipes, the velocity of the waves also depends on the properties of the solid skeleton of the conducting medium, first of all, on its compressibility. In addition, the speed of wave propagation in aquifers is significantly reduced in comparison with the speed of sound in water due to the process of destruction of the fluid structure.

So, in this subsection, a variant of the theory of relaxation filtration of a GW is proposed, which provides a consistent interpretation of the natural oscillations of the piezometric level (or pressure in a liquid) in a well that has exposed a pressure aquifer. At the same time, the characteristics of such oscillations (in particular, natural frequencies) are determined by the seepage parameters of the horizon, so there is no need to artificially introduce a water exchange condition in the well—aquifer system (Note 4). The proposed theory also allows theoretical studies of wave propagation patterns in the aquifer, which determine both free and forced oscillations of the piezometric level, and obtaining criterion estimates for such studies (some of these criteria are discussed above). It (this theory) provides the theoretical basis for the development of methods for conducting and interpreting EFT aquifers using methods based on initiating and tracking level fluctuations in a disturbing well. All of these tasks are discussed in the following sections.

2.2 Some Consequences of the Theory of Relaxation Filtration

Let us consider here some consequences of the proposed theory, which allow, in principle, to assess the possibility of creating methods for interpreting the results of the EFT on its basis, and to analyze the correctness of other theoretical concepts of the level fluctuation process.

The first and most important of them is a different, in comparison with the existing, idea of pressure waves (or pressure) caused by forced fluctuations in the level of GW in a perturbing well. Accordingly, from different positions, it is necessary to consider the wave effects that can occur in aquifers. Let us turn, in particular, to such an effect as filtration resonance. The fact is that, for example, the formal aspect of this problem was studied in [51]: the frequencies of the forced oscillations of the piezometric level in disturbing wells were justified, which should coincide with the natural frequencies and the resonance of these oscillations becomes likely, and the effect itself was accepted without any critical analysis as some objective reality. Meanwhile, the latter is not so obvious.

We turn first to the consideration of forced oscillations, i.e. fluctuations in the system, which acts a variable external field. Of particular interest is the case when the driving force is a simple periodic function of time with frequency γ, whence the pressure in the fluid at $r \to 0$ [20, 49]

$p(t) = B \cdot \cos \gamma t$,

where *B* is some constant (in our problem it is the amplitude of pressure fluctuations). It is convenient to present the last expression in a complex form, for which in its right part we write $\exp(i\gamma t)$ instead of $\cos \gamma t$:

$$p(t) = B \cdot \exp(i\gamma t) \tag{25}$$

With established forced oscillations, the energy of the system remains unchanged [36]; the system continuously absorbs (from a source of external force) energy dissipating due to the presence of friction. A manifestation of this is the constancy of the amplitude of pressure fluctuations p in the aquifer. It is easy to see – the equation (6) with regard to (9), (10) and (11) is converted to the form [20, 49]

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{2\beta}{c^2}\frac{\partial p}{\partial t} = 0 \tag{26}$$

whence it follows that under the assumption for forced oscillations the last term on the left side of (26) can be neglected:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 .$$

Obviously, it is completely analogous to how it occurs when pressure waves propagate in a fluid after pulsed excitation of an aquifer (these waves determine pressure oscillations with natural frequencies [20, 49]), the disturbance region and in the case of forced oscillations is limited and closed in terms of. When an aquifer is excited by an external force, a traveling cylindrical wave must come off the disturbing well. At its leading edge, the pressure gradient far exceeds the limiting one; however, as the wave advances, it flattens out, and at a certain distance *R* from the well axis the pressure gradient at the wave front becomes equal to the limiting one, the wave advancement stops, and it reflects inside the disturbed region.

In the course of the movement of such a wave, the structure of the liquid is completely destroyed. Its restoration, as noted above, is very slow compared to destruction. In addition, the supply of energy from the outside ensures the maintenance of the structure of the fluid in an energetically nonequilibrium (destroyed) state throughout the entire perturbation; therefore, it does not make sense to consider the conditions for the influence of the restoration of the structure on the laws of oscillations.

The imposition of radiated and reflected waves, as in the case of oscillations with its own frequencies, leads to the formation of standing waves; we confine ourselves here to the consideration of the steady-state process of pressure oscillations. Accordingly, the solution of the problem in the most general case should satisfy the following boundary conditions [46]:

$$\left.\begin{aligned} \alpha_1 p(0,t)+\alpha_2\frac{\partial p(0,t)}{\partial r}=0,\\ \delta_1 p(R,t)+\delta_2\frac{\partial p(R,t)}{\partial r}=0.\end{aligned}\right\} \quad (27)$$

We assume that there was no movement of water in the aquifer before the onset of the disturbance, then the initial pressure distribution in the horizon (initial condition) is given as [20, 49]

$p(r,t)=0,\ \dfrac{\partial p}{\partial t}=0$.

As mentioned earlier, the existence of a solid impermeable surface on the outer boundary of the perturbation region, at a distance *R* from its center, is reflected by condition $\dfrac{\partial p(R,t)}{\partial r}=0$ and $\nu=0$. The action of an external force causing forced oscillations of the piezometric level in the disturbing well is not accompanied by the extraction or injection of fluid into the horizon. In other words, in a standing cylindrical wave, *p* must remain finite at $r\to 0$ and $\dfrac{\partial p(0,t)}{\partial r}=0$. The above conditions completely satisfy (27), in which $\alpha_1=\delta_1=0$.

A nontrivial solution of the problem that satisfies conditions (27), as in all previous cases, when it is found by the method of separation of variables, is given in the most general form as the product [48]

$p(r,t)=T(t)\cdot f(r)$,

where $T(t)$ and $f(r)$ are arbitrary functions of time and coordinates.

In a monochromatic standing wave, the pressure distribution with regard to (25) is represented as [35]

$p(r,t)=\exp(i\gamma t)\cdot f(r)$, (28)

where for the function $f(r)$ we get

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \lambda^2 f = 0, \quad \lambda^2 = \frac{\gamma^2}{c^2}. \tag{29}$$

This is the zero order Bessel function equation. As mentioned above, in a standing cylindrical wave, *p* must remain finite at $r_0 \to 0$; the corresponding solution of (29) is $J_0(\lambda r)$—as before, the Bessel function of the first kind of order zero [1, 35].

Thus, (28) takes the form

$$p(r,t) = B \cdot \exp(i\gamma t) \cdot J_0(\lambda r).$$

We restrict ourselves to the real part, we have [20, 49]

$$p(r,t) = B \cdot \cos(\gamma t \pm \delta) \cdot J_0(\lambda r),$$

where δ is the phase angle.

To assess the relationship of the filtration parameters of the aquifer and the geometric characteristics of the disturbance region (its radius) with the frequency characteristics of pressure fluctuations, we use, as in the case of oscillations with natural frequencies, the condition on the outer boundary of this region:

$$\frac{\partial p(R,t)}{\partial r} = -B \cdot \cos(\gamma t \pm \delta) \cdot \lambda \cdot J_1(\lambda R) = 0,$$

whence $J_1(\lambda R) = 0$. The last equation has a countable set of roots $j_{1,n}$, i.e., $\lambda_n R = j_{1,n}$. Then, taking into account the second expression from (29), we obtain

$$\gamma = \frac{j_{1,n} \cdot c}{R} \tag{30}$$

Thus, forced pressure oscillations in a fluid (piezometric head) are described by the same system of equations as free oscillations, but with a zero oscillation damping factor β [13, 30].

We now return to the discussion of the possibility of resonance phenomena in an aquifer, in particular, pressure resonance in a fluid. Let us suppose that in a perturbing well, forced oscillations of the piezometric level are initiated with a given frequency and amplitude. In accordance with the solution given here, standing waves are formed in the aquifer in the region around the well with a certain radius *R* (determined by the limiting pressure gradient G_p or the limiting hydraulic gradient *G*). Suppose further - in the aquifer in the same area with the beginning of the disturbance (and under its influence) free pressure oscillations occur, coinciding in frequency with the forced, i.e. all prerequisites for pressure resonance are created [13, 30].

With the onset of resonance, the amplitude of pressure oscillations should start to grow rapidly, and immediately on all concentric circles inside the perturbation region corresponding to the pressure antinodes, including (and this is the most important in the process under consideration) and in the half-wave spatially adjacent to the outer boundary of the region, i.e. with $r = R$. The growth of the amplitude p in this half-wave should be accompanied by an increase in the pressure gradient $\left.\frac{\partial p}{\partial r}\right|_{r=R+0}$,

so that its value quickly becomes higher than the limiting G_p. Accordingly, the outer boundary of the disturbance region will inevitably move away from the disturbance center by a certain distance δR determined by condition $\left.\frac{\partial p}{\partial r}\right|_{r=(R+\delta R)+0} = G_p$ (note that the frequency of forced pressure fluctuations in fluid γ does not change by condition throughout the experiment, and the velocity of wave propagation in the aquifer does not change *c*).

And if so, then we should expect a change in the frequency of natural oscillations of pressure in the fluid (or piezometric level), which is rather rigidly associated with the geometrical parameters of the perturbation region (in particular, with its radius); this frequency, as follows from (30), will become equal to [20, 49]

$$\gamma = \frac{j_{1,n} \cdot c}{R + \delta R}.$$

The latter no longer coincides with the frequency of forced oscillations, therefore, the further development of pressure resonance in a fluid is no longer possible. In this case, all the additional energy (relative to the energy of the source of forced oscillations) released in the oscillatory system at the initial stage of resonance formation, went entirely to the destruction of the fluid structure in area δR.

In conclusion, let us dwell on one more consequence of the relaxation filtration of GW [20, 49]. Currently, there is only one field experimental determination of the limiting pressure gradient G_p or the limiting hydraulic gradient *G* according to the results of cluster pumping proposed by A.G. Arieh [3, 4]. However, our analysis of random errors in measurements and calculations arising from experiments and interpretation of their results clearly shows the inconsistency of this method, and, therefore, its inapplicability not only for parameter estimates of non-linear filtering law, but also for qualitative recording of the latter [15, 20].

You can also specify the method described in [32, 38], which is used in petroleum hydrogeology to identify additional filtration resistances in the bottomhole (to the disturbing well) zone, including those associated with the manifestation of various factors of non-stationary fluid filtration [32]. It is also used for the qualitative registration of the nonlinearity of the filtering law. At the same time, its implementation in practice for this purpose requires an obligatory study of the reliability and reliability of the conclusions obtained on its basis. It is in this part that the indicated method is clearly not developed. Accordingly, at present, the only theoretically reasonable and independent of measurement errors and calculations is the method of qualitative recording of nonlinearity of the filtration law at small pressure gradients, as already noted in [11, 20], a method of tracking oscillations of the piezometric level with natural frequencies after an impulse disturbance of the aquifer .

2.3 To the Method of Determining the Filtering Parameters of the Waterminal Horizons by the Results of the Tracking of Free and Forced Vibrations of the Piezometric Level of Underground Water

Estimations of filtration parameters of water-bearing sediments, performed on the basis of the interpretation of tracking free pressure fluctuations in a fluid (or piezometric) in a perturbing well, arising from the previously mentioned models I. Krauss and S.F. Grigorenko [8, 25, 26, 30, 31] is provided by taking into account water exchange in the well—aquifer system, although, as shown earlier, such a physical exchange is not provided for in principle by these physical and mathematical models. Let us try to estimate the relationship between the filtration parameters and the characteristics of free pressure oscillations in a fluid or the piezometric level of a GW based on taking into account such water exchange for a physical and mathematical model of GW relaxation filtration discussed above.

Note that water exchange between an aquifer and a well (determined by the difference in capacities of water-bearing sediments and a well) is not directly provided in the relaxation filtration model, since the kinematic inherently equation of this model describes the wave propagation in the aquifer – an isolated system. Formally, the oscillations of the piezometric level in a well are described by the same wave equation for the aquifer point, spatially coinciding with the well. On the other hand, there is no prohibition on such water exchange in the model, the latter is obviously determined by different values of well capacity and water-bearing sediments.

Let us proceed in the further presentation from the pressures in the formula (9) to the piezometric heads connected by the relation (1), so that the equation (9) takes the following form [14, 16, 20]:

$$H\ (r, t) = H^0 e^{-\beta t} \cdot \cos\left(\omega t \pm \phi\right) \cdot J_0\left(\kappa r\right) \tag{31}$$

where H^0 is the maximum difference in the piezometric head in the borehole with pulsed excitation of the aquifer.

Expressed in volumetric units, the GW inflow into the well (or outflow from it) is [14, 16, 20]

$$Q(t) = 2\pi T r_0 \left.\frac{\partial H\ (r, t)}{\partial r}\right|_{r=r_0} \tag{32}$$

Here T is the water conductivity of water-bearing sediments, $T = K\,m$ (here K is, as before, the filtration coefficient of water-bearing sediments, and m is the thickness of the aquifer); $H\left(r, t\right)$—as before, the change in the piezometric head in the aquifer at a distance r from the axis of the disturbing well at time t from the start of the experiment.

On the other hand, this inflow (outflow) can be determined by the rate of change of the level in the well [14, 16, 20]:

$Q(t) = \pi r_W^2 \ \dfrac{\partial H(t)}{\partial t}$, (33)

where r_W is the internal radius of the filter column (r_W is assumed to be constant along its entire length); $H(t)$ – change of piezometric level of GW in the well.

For simplicity, we neglect additional resistance to the imperfection of the well according to the nature of the opening of the horizon [16, 20]. Then $H(t) = H(r_0, t)$; equating the right-hand sides of relations (32) and (33), we have

$$2T r_0 \ \frac{\partial H}{\partial r} = r_W^2 \ \frac{\partial H}{\partial t} \tag{34}$$

Substitution in the last equation (34) of expression (31) gives for $\phi = 0$ [16, 20]

$$T = \frac{(\beta \cdot \cos \omega t + \omega \cdot \sin \omega t) \cdot r_W^2 \cdot J_0(\kappa r_0)}{2\kappa r_0 \cdot \cos \omega t \cdot J_1(\kappa r_0)},$$

or in a form convenient for further analysis, –

$$T = \frac{r_W^2 \cdot J_0(\kappa r_0)}{2\kappa r_0 \cdot J_1(\kappa r_0)} \left[\beta + \omega \frac{H^0 \cdot \sin \omega t}{H^0 \cdot \cos \omega t} \right] \tag{35}$$

Choosing on the graph of the time tracking of oscillations of the piezometric level the moments of time t_i so that $\omega t_i = \pi, 2\pi, \ldots,$ can be significantly simplified (35):

$$T = \frac{\beta r_W^2 \cdot J_0(\kappa r_0)}{2\kappa r_0 \cdot J_1(\kappa r_0)} \tag{36}$$

It is assumed that

$$H(r_0, t_i) = H^0 e^{-\beta t_i} \cdot J_0(\kappa r_0 \quad),$$

from where [16, 20]

$$J_0(\kappa r_0) = \frac{H(r_0, t_i)}{H^0} \ e^{\beta t_i} \tag{37}$$

From equation (37), using the tables of Bessel functions (given, for example, in [1]), it is easy to find the calculated values of the complex parameter κr_0 –

$\kappa r_0 = \inf J_0(\kappa r_0)$, (38)

and then functions $J_1(\kappa r_0)$. They, in accordance with (36), should provide an assessment of the water conductivity *T*.

Calculated from (38), taking into account the known r_0, the value of the parameter κ makes it possible, using equation (9), to determine the piezoconductivity of the aquifer, and, therefore, the capacity of water-bearing sediments:

$\mu^* = \frac{T}{\chi}$.

The solution of the problem of interpreting the experimental data shown here is actually obtained in neglecting the capacity of the well—this is precisely the condition (34) that reflects. At the same time, water exchange in the well—aquifer system, as it is presented above, makes sense for the relaxation filtration model only in the case of different capacities of water-bearing sediments and a well. And then the task of water exchange should be considered in a completely different formulation; in other words, the problem arises of taking into account such water exchange for assessing its effect on the reliability of the initial values determined from the experience, which provide calculations of filtration and capacity parameters of the aquifer [16, 20].

Consider this task. Let us assume that at the initial moment of the horizon excitation, the head values (and their changes relative to the initial value) in the well and the aquifer coincide. In the future, the piezometric pressure in the well throughout the whole EFT is in absolute value less than the pressure in the aquifer; its oscillations in the well occur with the same frequency as in the horizon, however, they lag in phase from the oscillations of the head in the latter (with the value of the phase angle ϕ_1). Then the water exchange condition (34) is written in the following form [16, 20]:

$$2Tr_0 \left. \frac{\partial H(r,t)}{\partial r} \right|_{r \to r_0} - 2Tr_0 \left. \frac{\partial \Delta h(r,t)}{\partial r} \right|_{r \to r_0} = r_W^2 \frac{\partial H(t)}{\partial t} - r_W^2 \frac{\partial \Delta h(t)}{\partial t} \quad (39)$$

where $\Delta h(r,t)$ and $\Delta h(t)$ are the differences between the theoretical and actual values of the head, respectively, in the aquifer and in the perturbing well. Obviously, in this ratio $\Delta h(r_0,t) = \Delta h(t)$.

Take the simplest case. Let's set $\Delta h(r_0,t)$ in the following form [16, 20]:

$$\Delta h(r_0,t) = \Delta h^0 \left[\left(1 - e^{-\beta_1 t}\right) e^{-\beta t} \right] \cdot \cos(\omega t - \phi_1) \cdot J_0(\kappa r_0) \quad (40)$$

where Δh^0 is the initial (maximum) value of the difference between the theoretical and actual values of pressure in the aquifer and in the well. She is in the process of indignation "typed" almost instantly, so that $\beta_1 >> \beta$.

Taking into account the last condition, and, also remembering that steady-state oscillations of the piezometric level are considered, simplify expression (40):

$$\Delta h(r_0,t) = \Delta h^0 e^{-\beta t} \cdot \cos(\omega t - \phi_1) \cdot J_0(\kappa r_0) \quad (41)$$

Substitution in (39) of the expression from (31) and equation (41) gives [16, 20]

$$T = \frac{r_W^2 \cdot J_0(\kappa r_0)}{2\kappa r_0 \cdot J_1(\kappa r_0)} \left[\beta + \omega \frac{H^0 \cdot \sin(\omega t - \phi) - \Delta h^0 \cdot \sin(\omega t - \phi_1)}{H^0 \cdot \cos(\omega t - \phi) - \Delta h^0 \cdot \cos(\omega t - \phi_1)} \right] \quad (42)$$

It is easy to see that, with $\phi_1 = \phi = 0$ and $\omega t_i =$ π, 2π, … the last expression degenerates into (36). On the basis of (42), it becomes possible to estimate the systematic error of the model for the value of the transmissibility *T* established from experience, due to the influence of the perturbation well capacity. We put in it, as in (35), $\phi = 0$ and represent for the moments of time that meet condition $\omega t_i =$ π, 2π, … in the following form:

$$T = \frac{r_W^2 \cdot J_0(\kappa r_0)}{2\kappa r_0 \cdot J_1(\kappa r_0)} \left[\beta - \omega \frac{\Delta h^0 \cdot \sin(n\pi - \phi_1)}{H^0 - \Delta h^0 \cdot \cos(n\pi - \phi_1)} \right] \tag{43}$$

where is $n = 1, 2, \ldots.$.

Obviously, one of the components of this error is completely determined by the value of the subtracted in the right-hand side of (43), and that, in turn, depends on the ratio $\dfrac{\Delta h^0}{H^0}$ and the phase angle ϕ_1 [16, 20]:

$$\Psi\left(H^0, \Delta h^0, \phi_1\right) \equiv \frac{\Delta h^0 \cdot \sin(n\pi - \phi_1)}{H^0 - \Delta h^0 \cdot \cos(n\pi - \phi_1)} = \frac{a \cdot \sin(n\pi - \phi_1)}{1 - a \cdot \cos(n\pi - \phi_1)}, \quad a = \frac{\Delta h^0}{H^0}.$$

Calculated for a number of values of *a*, ϕ_1 and *n* values of this component of the systematic error of the model are given in Table 1 and Fig. 3 [16, 20]. As it follows from the graphs presented in this figure, it rapidly grows with an increase in the phase angle ϕ_1 and the difference between the capacities of the well and water-bearing sediments, expressed in the growth of the parameter *a*. In addition, the component was calculated only for values of *n* equal to 1 and 2. For large values of *n*, the values of the function $\Psi\left(H^0, \Delta h^0, \phi_1\right)$ are exactly equal to those shown in Fig. 3 and in Table 1.

For example, at $\beta = 0.08$ s^{-1}, $\omega = 0.4$ s^{-1} and $n = 2$, the phase angle ϕ_1 is 0.6 rad and the ratio $\dfrac{\Delta h^0}{H^0}$ from 0.1 to 0.5 determines the relative systematic error of the conductivity *T* from 49.7 to 240.5 %.

The task of interpreting the results of tracking the free oscillations of the piezometric level of the GW in the well is thus reduced to identifying and eliminating the indicated systematic error, or otherwise, to identifying the component of these oscillations that corresponds to the theoretical solution. Of fundamental importance for the solution of

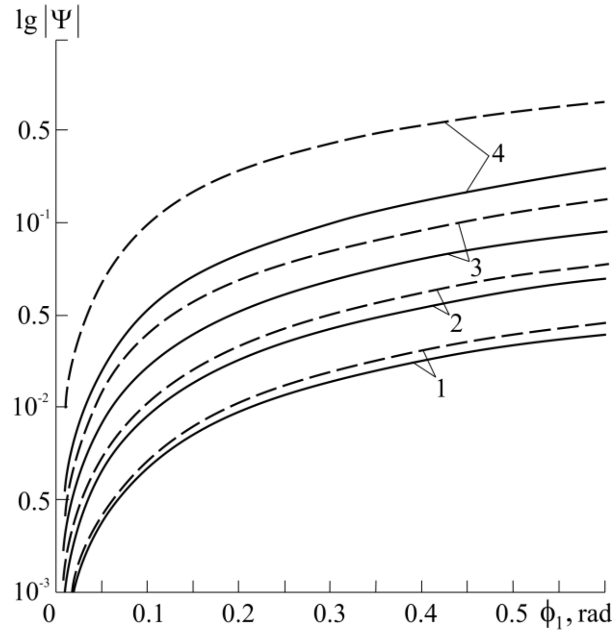

Figure 3. Graphs of the Dependence of the Error Function on the Phase Shift for the Values of the Parameter *a*, Equal to 0.05 (1), 0.1 (2), 0.2 (3) and 0.5 (4), Respectively, and for *n* = 1 (Solid Line) and *n* = 2 (Dashed Line)

Table 1. The Values of the Function $\Psi\left(H^0, \Delta h^0, \phi_1\right)$ Depending on the Values of the Phase Angle ϕ_1, the Parameter $a = \dfrac{\Delta h^0}{H^0}$ and the Multiplier *n*

ϕ_1	*n* = 1				*n* = 2			
	a = 0.05	*a* = 0.1	*a* = 0.2	*a* = 0.5	*a* = 0.05	*a* = 0.1	*a* = 0.2	*a* = 0.5
0.01	0.0004762	0.0009091	0.0016667	0.0033333	-0.0005263	-0.0011111	-0.0024999	-0.0099993
0.05	0.0023801	0.0045441	0.0083316	0.0166667	-0.0026303	-0.0055525	-0.0124909	-0.0499168
0.10	0.0047551	0.0090799	0.0166528	0.0333333	-0.0052530	-0.0110864	-0.0249272	-0.0993371
0.15	0.0071199	0.0135992	0.0249531	0.0499999	-0.0078605	-0.0165835	-0.0372549	-0.1477787
0.20	0.0094694	0.0180936	0.0332219	0.0666661	-0.0104453	-0.0220256	-0.0494211	-0.1947866
0.25	0.0117986	0.0225550	0.0414488	0.0833315	-0.0130000	-0.0273947	-0.0613740	-0.2399447

0.30	0.0141024	0.0269750	0.0496228	0.0999955	-0.0155172	-0.0326734	-0.0730642	-0.2828855
0.35	0.0163757	0.0313453	0.0577330	0.1166568	-0.0179898	-0.0378448	-0.0844445	-0.3232972
0.40	0.0186137	0.0356576	0.0657683	0.1333140	-0.0204109	-0.0428925	-0.0954705	-0.3609271
0.45	0.0208113	0.0399035	0.0737174	0.1499650	-0.0227736	-0.0478008	-0.1061007	-0.3955840
0.50	0.0229637	0.0440746	0.0815685	0.1666070	-0.0250714	-0.0525547	-0.1162972	-0.4271366
0.55	0.0250659	0.0481627	0.0893097	0.1832367	-0.0272980	-0.0571401	-0.1260254	-0.4555106
0.60	0.0271132	0.0521594	0.0969287	0.1998497	-0.0294473	-0.0615437	-0.1352546	-0.4806841

such a problem, as already noted above, is the possibility of a reliable estimate of the damping factor β for the oscillations from experimental data. We investigate this problem in more detail.

Let us choose on the graph of tracking the maxima and minima of the values of the change in the piezometric level in the well $H_W(t)$. We subtract equation (40) from (31) and write the resulting expression for two such neighboring maxima (two adjacent minima). Given that $\beta_1 >> \beta$, this expression can be represented as [16, 20]

$$H_W(t_1) = e^{-\beta t_1}\left[H^0 \cdot \cos\omega t_1 \cdot J_0(\kappa r_0) - \Delta h^0 \cdot \cos(\omega t_1 - \phi_1) \cdot J_0(\kappa r_0)\right] \equiv F_1\, e^{-\beta t_1},$$

$$H_W(t_2) = e^{-\beta t_2}\left[H^0 \cdot \cos\omega t_2 \cdot J_0(\kappa r_0) - \Delta h^0 \cdot \cos(\omega t_2 - \phi_1) \cdot J_0(\kappa r_0)\right] \equiv F_2\, e^{-\beta t_2}.$$

Here all the designations are the same.

Take the natural logarithms of the last two ratios, we have

$$\ln H_W(t_1) = -\beta t_1 + \ln F_1 \tag{44}$$

$$\ln H_W(t_2) = -\beta t_2 + \ln F_2 \tag{45}$$

Due to the fact that $\cos\omega t_2 = \cos(\omega t_1 + \omega\Delta t)$ and $\cos(\omega t_2 - \phi_1) = \cos(\omega t_1 + \omega\Delta t - \phi_1)$, and $\omega\Delta t = 2\pi$, in equations (44) and (45) should be $F_1 = F_2$. Then, as a result of subtracting (45) from (44) it follows [16, 20]

$$\beta = \frac{\ln \dfrac{H_W(t_1)}{H_W(t_2)}}{t_2 - t_1}.$$

Thus, the difference in the capacity of the disturbing well from the capacity of the aquifer does not affect the reliability of the estimates of the oscillation damping coefficient from experimental data. Accordingly, there is a simple possibility of restoring the piezometric head values in the well at any time point during the disturbance, corresponding to the theoretical solution, and, therefore, at first glance, the above considered method for determining the filtration parameters of the aquifer becomes applicable.

Indeed, with the measured initial value of the pressure drop in the well $H_W(0) = H^0 \cdot J_0(\kappa r_0)$ and the magnitude of the attenuation coefficient of the oscillations β, it is easy to calculate $H(r_0, t)$:

$$H(r_0, t) = H^0 e^{-\beta t} \cdot \cos\omega t \cdot J_0(\kappa r_0) \quad (46)$$

There is no particular difficulty in determining, on the basis of experimental data, such disturbance parameters as the oscillation frequency ω and the duration of the decay of the shear stresses in the liquid τ_0.

At the same time, nothing has been said about the physical sense and the methods for estimating parameter H^0. In particular, it follows from the relaxation filtration model (and this confirmed the study of the effect of well capacity on the reliability of filtration parameters) that H^0 really should be not just the value of the pressure drop in the well at the time of the beginning of the disturbance, but the magnitude of this difference on the axis of the disturbing well. In this case, the latter, as is supposed theoretically, should have a capacity that coincides with the capacity of the aquifer.

Accordingly, a theoretical change in the piezometric head at a distance r_0 from the borehole axis is established using formula (46) using the measured complex parameter $H_W(0) = H^0 \cdot J_0(\kappa r_0)$, in which the H^0 value, in turn, cannot be directly measured, and its calculation is based on the values (measured and calculated by experienced data) $H_W(0)$, ω, β and τ_0 also requires an a priori task, as follows from equation (10), the piezoconductivity (Note 5) χ. Using the value of H^0 established from experience as $H_W(0)$ requires accepting condition $J_0(\kappa r_0) = 1$ or, which is the same thing, $\kappa r_0 = 0$. This, in turn, means $J_1(\kappa r_0) = 0$, and then the value of transmissibility *T* in equation (36) [16, 20] becomes uncertain.

The correctness of the water exchange scheme discussed above, i.e. compliance with its real conditions should be reflected in these EFT; first of all, this refers to recording phase angle ϕ_1 in the well of piezometric level fluctuations in the well. The possibility of estimating the difference between the theoretical and actual head values in well $\Delta h(t)$ without the results of tracking the piezometric head oscillations (or pressure in the fluid) in the aquifer at a certain distance from the axis A disturbing well seems at least problematic. So, in particular, the first and second quarter periods of the period (respectively, the first and second half periods should be different) should be fixed on the oscillation tracking chart over the first half period, and the first period should differ from the following (equal to each other) as times for the very difference of the first two quarters of the period.

As noted earlier, in Fig. 1 shows an indicator diagram of tracking free oscillations of a piezometric level in a disturbing well after pulsed excitation of an aquifer. Processing it gave the following value of the oscillation frequency – $\omega_W \cong 0.448$ s^{-1} (the "W" index, as before, denotes the parameters set for the disturbing well), respectively, the value of the oscillation period T ≅ 15.2 s. Hence, the theoretically calculated value of a quarter of a period is 3.8 s, however, from the diagram, this value is determined to be approximately 3.3 s. Thus, the difference of the first quarters of the first period is about 0.5 s, respectively, $\phi_1 \cong 0.207$ rad.

3. Result

So, the study performed showed that, firstly, there is no basic perceived advantage associated with the possible use of methods for interpreting the results of tracking free and forced vibrations of the piezometric level: the capabilities of the EFT aquifer with a single production (well) and its short duration. Secondly, the technical difficulties encountered in organizing such testing and interpreting its results are currently insurmountable. Therefore, the implementation of methodological developments based on the theory of relaxation filtration is possible and expedient only on fundamentally different, in comparison with existing, technical and methodical approaches to the experimental study of wave propagation in an aquifer.

4. Discussion

Thus, the considered method, based on the model of relaxation filtration, does not actually determine the filtration parameters of the aquifer based on the interpretation of the tracking of free oscillations of the piezometric level in a disturbing well. The situation could be corrected if we add the method by determining, for example, the distance from the center of the perturbation to the nodes of cylindrical standing waves corresponding to the zeros of function $J_0(\kappa r)$, in order to establish the parameter κ and accordingly the value H^0. To do this, it is necessary to drill an observation well and its equipment that would completely eliminate the effect of capacity on the tracking results (for example, tensiometers, reservoir pressure sensors).

However, in reality, the pressure drop in a perturbing well is set to no more than 10, maximum 15 centimeters [30, 31], in the observation well it is even smaller, even if the well is at the point of the wave antinode. This places extremely high demands on the sensitivity and accuracy of instrumentation. In the experiments of S. F. Grigorenko, the value of $H_W(0)$ was set to 8 and even 15 m [8, 25, 26], respectively, the requirements for accuracy and sensitivity of the measuring equipment are reduced. At the same time, in both cases there are no criteria providing the choice of the distance from the disturbance center at which the observation well should be drilled, i.e. it is uncertain whether this well is associated (even if it does not coincide with a node or an antinode of a standing wave) to some point of the standing wave profile. By the way, it should be noted that exactly the same requirements are imposed on the equipment of observation wells for carrying out and interpreting the results of the EFT aquifer when it initiates forced oscillations of a piezometric level with a small amplitude, which are formally described by the same system of equations (9), (10) and (11), but with attenuation coefficient $\beta = 0$ [20, 49].

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Notes

Note 1. Since the basis of the physical and mathematical filtering model considered in this subsection is based on the con-cepts of relaxation phenomena, the model itself is designated by us, followed by Yu.M. Molokovich [39], as "a model (or theory) of relaxation filtration". This is also reflected in the title of this work.

Note 2. Let us leave aside the discussion of the technical possibilities of such a perturbation through a horizontal drainage of considerable length perfect for the degree of reservoir opening (which revealed the pressure bed). We will be interested only in a formal representation of the process of oscillations of a piezometric level with its own frequencies.

Note 3. Of course, such a wave is flattened out, but only due to energy dissipation. At some sufficiently large distance from the internal contour of the perturbation, the pressure gradient at the wave front becomes equal to the limiting one; however, the energy of the reflected wave (including due to dissipation in the disturbed region) is already insufficient for the formation of standing waves. Similar phenomena in one degree or another manifest themselves in cylindrical and spherical waves, which is reflected, in particular, in the degree of damping of oscillations.

Note 4. Of course, if the capacities of the well and water-bearing sediments are the same, otherwise this condition deter-mines the internal boundary condition and must be introduced.

Note 5. In the same way, the a priori assignment of the piezoelectric conductivity χ requires the method of interpreting the results of the tracing. S.F. Grigorenko [8, 25, 26].